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## A Classification of Semi-simple Rings

by

A. SULIŃSKI

*Presented by A. MOSTOWSKI on October 4, 1960*

This paper is a direct continuation of [4] and like that paper, is concerned with the investigation of rings semi-simple in the sense of the Brown-McCoy radical [3] (i.e. subdirect sums of simple rings with unity). First there are considered the homomorphisms of a semi-simple ring modulo its special and completely non-special part. Then a transfinite process is defined, which permits us to assign, in the unique way, to each semi-simple ring an ordinal, which is called the type of the ring. It is proved that for a strongly semi-simple ring of the type  $\tau$  we can uniquely define a composition sequence (also of the type  $\tau$ ), which consists of strongly semi-simple rings decomposable into the direct sum of the special and completely non-special part. Finally, some properties of such a sequence are investigated.

1. Let  $r$  be a semi-simple ring and let  $M$  be its structure space [4]. Therefore,  $r$  can be subdirectly imbedded into the complete direct sum  $\sigma(M)$  of simple rings  $r_m = r/m$  ( $m \in M$ ) with unity. Each element  $x \in r$  can be regarded as a function  $x(m)$  defined for  $m \in M$  and such that  $x(m) \in r_m$ . Let us denote by  $h_N$  ( $N \subseteq M$ ) such a mapping of  $r$ , that each  $x \in r$  is transformed on the function  $\bar{x}(m)$  defined as follows: for  $m \in N$  we put  $\bar{x}(m) = x(m)$ , and for  $m \in M - N$  we put  $\bar{x}(m) = 0$ . The mapping  $h_N$  is obviously a homomorphism of  $r$  and the ideal  $I(N)$  [4] is the kernel of the homomorphism.

LEMMA 1. *The ring  $\bar{r} = r/I(N)$  can be subdirectly imbedded into the direct sum  $\sigma(N)$ .*

Proof. The homomorphism  $h_N$  maps the ring  $r$  onto some subring  $r'$  of  $\sigma(N)$ . But the ideal  $I(N)$  is the kernel of  $h_N$ , hence  $r'$  is isomorphic with  $\bar{r}$ . Thus,  $\bar{r}$  is subdirectly imbedded into  $\sigma(N)$  ([4], Def. 1.).

LEMMA 2.  *$I_N(N_1) = I(N_1)/I(N)$ , where  $N_1 \subset \bar{N}$ ,  $N \subseteq M$  and  $I_N(N_1)$  is the intersection of such ideals  $m/I(N)$  of  $\bar{r} = r/I(N)$  that  $m \in N_1$ .*

Proof. Let us assume that  $\bar{x} \in I_N(N_1)$ . Then  $\bar{x}(m) = 0$  for each  $m \in N_1 \subseteq \bar{N}$ , since  $\bar{r}$  is subdirectly imbedded into  $\sigma(N)$ . Hence, a certain counter-image  $x$  of the element  $\bar{x}$  under the homomorphism  $h_N$  of  $r$  onto  $\bar{r}$  belongs to  $I(N_1)$ . Thus,  $\bar{x} \in I(N_1)/I(N)$ . Conversely, let us suppose that  $\bar{x} \in I(N_1)/I(N)$ . Then  $x \in I(N_1)$  and  $\bar{x}(m) = 0$  for each  $m \in N_1$ . Thus,  $\bar{x} \in I_N(N_1)$ .

Let us denote by  $\varrho(M)$  the discrete direct sum of the rings  $r_m (m \in M)$ .

DEFINITION 1. Let  $r_1, r_2$  be two rings subdirectly imbedded into  $\sigma(M)$ . We shall say that these rings are subdirectly imbedded in similar position into  $\sigma(M)$ , if  $\varrho(M) \cap r_1 = \varrho(M) \cap r_2$ .

In particular, for the completely non-special ring  $r$  subdirectly imbedded into  $\sigma(M)$  we have  $\varrho(M) \cap r = 0$  ([4], Th. 7). Thus all completely non-special rings subdirectly imbedded into  $\sigma(M)$  are imbedded in a similar position.

LEMMA 3. Let  $r_1, r_2$  be two special (completely non-special) rings subdirectly imbedded in similar position into  $\sigma(M)$ . Let  $r$  be a ring subdirectly imbedded into  $\sigma(M)$ . Then from  $r_1 \subseteq r \subseteq r_2$  it follows that  $r$  is also special (completely non-special) and  $r$  is subdirectly imbedded into  $\sigma(M)$  in a similar position as  $r_1$  and  $r_2$ .

Proof. We have  $r_1 \cap \varrho(M) \subseteq r \cap \varrho(M) \subseteq r_2 \cap \varrho(M)$ . But  $r_1 \cap \varrho(M) = r_2 \cap \varrho(M)$ , since  $r_1$  is subdirectly imbedded into  $\sigma(M)$  in a similar position as  $r_2$ . Hence,  $r \cap \varrho(M) = r_1 \cap \varrho(M) = r_2 \cap \varrho(M)$ . Thus,  $r$  is imbedded into  $\sigma(M)$  in a similar position as  $r_1$  and  $r_2$ .

Let us denote by  $D, D_1, D_2$  the minimal sets of  $r, r_1, r_2$ . From the fact that  $r \cap \varrho(M) = r_1 \cap \varrho(M) = r_2 \cap \varrho(M)$  we obtain that  $m \in D$  (i.e.  $(m)^* \neq 0$ , [4], Lem. 4 and 5) if and only if  $m_1 \in D_1$  and  $m_2 \in D_2$ , where  $m, m_1, m_2$  are maximal modular ideals of  $r, r_1, r_2$  such that  $m_1 \subseteq m \subseteq m_2$  and  $r_1/m_1 \sim r/m \sim r_2/m_2 \sim r_m$ . If the rings  $r_1, r_2$  are completely non-special then  $D_1 = D_2 = \emptyset$ , hence  $D = \emptyset$  and  $r$  is also completely non-special ([4], Def. 6). If, however,  $r_1, r_2$  are special, then  $I_1(D_1) = I_2(D_2) = 0$ , where  $I_i(D_i)$  ( $i = 1, 2$ ) is the intersection of all  $m_i \in D_i$ . Then  $I_1(D_1) \subseteq I(D) \subseteq I_2(D_2)$ , since  $m_1 \subseteq m \subseteq m_2$ . Hence,  $I(D) = 0$  and  $r$  is also special.

THEOREM 1. Let  $r$  be a semi-simple ring, let  $M$  be its structure space and let  $D$  be the minimal set of  $r$ . Then the ring  $r' = r/n(r)$  is special and can be subdirectly imbedded into  $\sigma(\text{int } \bar{D})$  in similar position as  $s(r)$ ; the ring  $r'' = r/s(r)$  is completely non-special and can be subdirectly imbedded into  $\sigma(M - \bar{D})$ , where  $s(r)$  is the special part,  $n(r)$  — completely non-special part of  $r$  ([4], Def. 7) and  $\text{int } \bar{D} = M - \overline{M - D}$ .

Proof. By the Lemma 1, the ring  $r'$  can be subdirectly imbedded into  $\sigma(\text{int } \bar{D})$ , and the ring  $r''$  — into  $\sigma(M - \bar{D})$ , since  $n(r) = I(\bar{D}) = I(\text{int } \bar{D})$  ([4], Def. 7 and Th. 8) and  $s(r) = \sigma(M - \bar{D})$ . But, on the other hand,  $s(r)$  can be subdirectly imbedded into  $\sigma(\text{int } \bar{D})$  and  $n(r)$  can be subdirectly imbedded into  $\sigma(M - \bar{D})$  ([4], Lem. 1).

Now, we shall prove that  $r'$  is subdirectly imbedded into  $\sigma(\text{int } \bar{D})$  in similar position as  $s(r)$ . Indeed, if  $m^* \neq 0$ , then  $m \in D \subseteq \text{int } \bar{D}$  ([4], Lemmas 5 and 7). Hence  $[m/n(r)]^* \neq 0$ . Conversely, let us suppose that  $[m/n(r)]^* \neq 0$  where  $m \supseteq n(r) = I(\bar{D})$ , i.e.  $m \in \bar{D}$ . Then, by the Lemma 2 and ([4], Lemma 2), we have  $[m/n(r)]^* = [m/I(\bar{D})]^* = I_{\bar{D}}(\bar{D} - (m))$ . Let  $\bar{x} \in I_{\bar{D}}(\bar{D} - (m))$ ,  $\bar{x} \neq 0$ . Then a certain counter-image  $x$  of the element  $\bar{x}$  under the homomorphism  $h_{\bar{D}}$  belongs to  $I(\bar{D} - (m))$ . If  $m \notin \text{int } \bar{D}$ , then  $\bar{D} - (m) \supseteq \text{int } \bar{D}$ . Then  $x \in I(\bar{D} - (m)) \subseteq I(\text{int } \bar{D}) = I(\bar{D}) = n(r)$  ([4], Th. 8), what is impossible, since  $\bar{x} \neq 0$ . Thus,  $m \in \text{int } \bar{D}$ . But, on the

other hand, there exists such  $y \in s(r)$  that  $y(m) = 1_m \in r_m$  since the ring  $s(r) = I(M - \bar{D})$  is subdirectly imbedded into  $\sigma(\text{int } \bar{D})$  ([4], Lem. 1). Therefore,  $xy(m) = \bar{x} \cdot 1_m \neq 0$  and  $xy \in I(\bar{D} - (m)) \cap I(M - \bar{D}) = I(M - (m)) = m^*$  ([4], Coroll. 3). Thus,  $m^* \neq 0$ .

Now, we shall prove that the ring  $r'$  is special. Indeed, from the fact proved above we derive that  $D$  is also the minimal set of  $r'$ . Hence, by Lemma 2, we have  $I_{\text{int } \bar{D}}(D) = I(D)/I(\text{int } \bar{D}) = I(\bar{D})/I(\bar{D}) = 0$ . Thus,  $r'$  is special.

Finally, we shall prove that the ring  $r''$  is completely non-special. Indeed, let us suppose, that there exists such  $m \in M - \bar{D}$  that  $[m/s(r)]^* \neq 0$ . Let  $\bar{x} \in [m/s(r)]^* = I_{M-\bar{D}}((M - \bar{D}) - (m))$ ,  $\bar{x} \neq 0$ . Then a certain counter-image  $x$  of the element  $\bar{x}$  under the homomorphism  $h_{(M-\bar{D})}$  belongs to  $I((M - \bar{D}) - (m))$ . But, on the other hand, there exists such  $y \in n(r) = I(D)$  that  $y(m) = 1_m \in r_m$ , since  $n(r)$  is subdirectly imbedded into  $\sigma(M - \bar{D})$ . Then  $xy(m) = \bar{x} \cdot 1_m \neq 0$  and  $xy \in I((M - D) - (m)) \cap I(\bar{D}) = I(M - (m)) = m^*$ . Thus  $m^* \neq 0$  what is impossible because of  $m \in M - \bar{D}$  ([4], Lem. 5).

2. Let  $r$  be a semi-simple ring, let  $M$  be its structure space and let  $D$  be the minimal set of  $r$ . Let  $M'$  be the derivative of the space  $M$ , i.e. the set of such  $m \in M$ , that  $m \in \overline{M - (m)}$  (or, in other words,  $m^* = 0$ ). Then  $D = M - M'$ . We put  $D_1 = D$ ,  $M_0 = M$ ,  $M_1 = \text{Fr } \bar{D}_1$ , where  $\text{Fr } \bar{D}_1 = \bar{D}_1 \cap M - \bar{D}_1$ . Let us suppose that for each ordinal  $\xi < \alpha$  the sets  $D_\xi$  and  $M_\xi$  are defined. If there exists the predecessor ordinal  $\alpha - 1$ , then we put  $D_\alpha = M_{\alpha-1} - M'_{\alpha-1}$ , where  $M'_{\alpha-1}$  is the derivative of the set  $M_{\alpha-1}$  (i.e. the set of such  $m \in M$  that  $m \in \overline{M_{\alpha-1} - (m)}$ ). Furthermore, we put  $M_\alpha = \bar{D}_\alpha \cap M_{\alpha-1} - (\bar{D}_\alpha)$ . If however,  $\alpha$  is a limit ordinal, then we put  $D_\alpha = M_\alpha - M'_\alpha$ , where  $M_\alpha = \bigcap_{\xi < \alpha} M_\xi$  and  $M'_\alpha$  is again the derivative of the set  $M_\alpha$ . We have just defined two descending sequences of sets  $\bar{D}_1 \supset \bar{D}_2 \supset \dots$  and  $M_0 \supset M_1 \supset \dots$ . Thus, there exists such an ordinal  $\tau$ , that  $M_\tau = M_{\tau+1}$ . Then  $M_{\tau+1} = \bar{M}_\tau - \bar{D}_{\tau+1} \cap \bar{D}_{\tau+1} = M_\tau$ . Hence, by the fact that  $\bar{D}_{\tau+1} \subseteq M_\tau$ , we obtain  $\bar{D}_{\tau+1} = M_\tau$ , i.e.  $M_\tau = \emptyset$ .

Note that all the sets  $M_\alpha$  ( $\alpha < \tau$ ) are closed.

DEFINITION 2. The least ordinal  $\tau$  such that  $M_\tau = \emptyset$  will be called the type of the semi-simple ring  $r$ .

Observe that, by the procedure described above, we obtain immediately from Lemma 1:

COROLLARY 1. Let  $r$  be a semi-simple ring of a type  $\tau$ . Then for each  $\alpha < \tau$  the set  $M_\alpha$  is the structure space of  $r_\alpha = r/I(M_\alpha)$  and  $D_{\alpha+1}$  is the minimal set of  $r_\alpha$ .

From Theorem 1 and Corollary 1 we obtain

COROLLARY 2. For each  $\alpha < \tau$  the ring  $r'_\alpha = r_\alpha/n(r_\alpha)$  is special and can be subdirectly imbedded into  $\sigma(\text{int } \bar{D}_{\alpha+1})$  in a similar position as  $s(r_\alpha)$ , where  $\text{int } \bar{D}_{\alpha+1} = M_\alpha - M_\alpha - \bar{D}_{\alpha+1}$ . Furthermore, the ring  $r''_\alpha = r_\alpha/s(r_\alpha)$  is completely non-special and can be subdirectly imbedded into  $\sigma(M_\alpha - \bar{D}_{\alpha+1})$ .

LEMMA 4. Let  $r$  be a strongly semi-simple ring ([4], Def. 4) and let  $M$  be its structure space. Then  $I(N_1) + I(N_2) = I(\bar{N}_1 \cap \bar{N}_2)$ , where  $N_1, N_2 \subseteq M$ .

Proof. By the strong semi-simplicity of  $r$  we obtain  $I(N_1) + I(N_2) = I(N)$ , where  $N \subseteq M$ . Let us suppose that  $m \in \bar{N}$ . Then  $m \supseteq I(N) = I(N_1) + I(N_2)$ , hence  $m \supseteq I(N_1)$  and  $m \supseteq I(N_2)$ , i.e.  $m \in \bar{N}_1$  and  $m \in \bar{N}_2$ . Thus  $m \in \bar{N}_1 \cap \bar{N}_2$ . Conversely, let us suppose that  $m \in \bar{N}_1 \cap \bar{N}_2$ . Then  $m \supseteq I(N_1)$  and  $m \supseteq I(N_2)$ , hence  $m \supseteq I(N_1) + I(N_2) = I(N)$ . Thus  $m \in \bar{N}$ .

THEOREM 2. If  $D$  is the minimal set of a strongly semi-simple ring  $r$  then  $I(\text{Fr } \bar{D}) = s(r) \dot{+} n(r)$ , where  $\text{Fr } \bar{D} = \bar{D} \cap M - \bar{D}$ .

Proof. Let us notice that  $s(r) = I(M - \bar{D})$  and  $n(r) = I(\bar{D})$  ([4], Def. 7). Hence, by Lemma 4, we have  $s(r) \dot{+} n(r) = I(\bar{D} \cap M - \bar{D}) = I(\text{Fr } \bar{D})$ . The sum  $s(r) \dot{+} n(r)$  is direct, since  $s(r) \cap n(r) = I(M - \bar{D}) \cap I(\bar{D}) = I(M) = 0$ .

LEMMA 5. Each ideal of a strongly semi-simple ring is (as a ring) also strongly semi-simple.

Proof. \*) Let  $a$  be an ideal of a strongly semi-simple ring  $r$  and let  $b$  be an ideal of  $a$ . Note that the ring  $r$  is strongly semi-simple if and only if each of its homomorphic images is semi-simple (cf. [1]). Thus,  $a^2 = a$  because, in the opposite case, the semi-simple ring  $r/a^2$  would contain the nilpotent ideal  $a/a^2$ , what is impossible. Let us denote by  $(b)_r$  the ideal of  $r$  generated by the set  $b$ . Then  $(b)_r^3 \subseteq b$  since  $(b)_r^3 \subseteq a(b)_r$ ,  $a \subseteq a(b + br + r\bar{b} + r\bar{b}r) \subseteq aba + aba + aba + aba \subseteq b$ , (cf. [2], Lem. 4). Hence, we have  $(b)_r^3 \subseteq b \subseteq (b)_r$  and  $(b)_r = b$ , since  $(b)_r^2 = (b)_r$ . Thus,  $b$  is also an ideal of  $r$ , and, by the strong semi-simplicity of  $r$ ,  $b = I(N)$ ,  $N \subseteq M$ . Furthermore  $a = I(N_1)$ ,  $N_1 \subseteq M$ , where  $N \supset N_1$ , since  $b \subseteq a$ . But  $a$  is subdirectly imbedded into  $\sigma(M - \bar{N}_1)$  ([4], Lem. 1). Hence, each ideal  $a \cap m$ ,  $m \in M - \bar{N}_1$  is maximal modular in  $a$ . Then the intersection of all ideals  $a \cap m$ ,  $m \in \bar{N} \cap M - \bar{N}_1$  is  $I(\bar{N} \cap (M - \bar{N}_1)) \cap a = I(\bar{N} \cap (M - \bar{N}_1)) \cap I(\bar{N}_1) = I((\bar{N} \cap (M - \bar{N}_1)) \cup \bar{N}_1) = I(\bar{N}) = b$ . Thus,  $b$  is representable ([4], Def. 2 and 4).

DEFINITION 3. The semi-simple ring  $r$  will be called *decomposable*, if  $r$  is the direct sum of its special and completely non-special parts.

Notice that, if a strongly semi-simple ring  $r$  has type 1, then  $r$  is decomposable. Indeed, in this case  $M_1 = \text{Fr } \bar{D} = \emptyset$  and, by Theorem 2,  $r = I(\emptyset) = I(\text{Fr } \bar{D}) = s(r) \dot{+} n(r)$ .

THEOREM 3. Let  $r$  be a strongly semi-simple ring of type  $\tau$ . Then for each  $\alpha < \tau$  the ring  $c_\alpha = I(M_{\alpha+1})/I(M_\alpha)$  is decomposable and strongly semi-simple.

Proof. From Corollary 1 we have that  $M_\alpha$  is the structure space of  $r_\alpha = r/I(M_\alpha)$  and  $D_{\alpha+1}$  is the minimal set of  $r_\alpha$  ( $\alpha < \tau$ ). Hence, by Lemma 2, applied to the ring  $r_\alpha$ , we obtain  $c_\alpha = I(M_{\alpha+1})/I(M_\alpha) = I_{M_\alpha}(M_{\alpha+1})$ , where  $M_{\alpha+1} = \bar{D}_{\alpha+1} \cap M_\alpha - \bar{D}_{\alpha+1}$ . Let us observe that each homomorphic image of a strongly semi-simple ring is also strongly semi-simple [1]. Then, applying Theorem 2 to the strongly

\*) This proof was kindly communicated to me by Dr E. Sasiada.

semi-simple ring  $r_\alpha$ , we obtain  $c_\alpha = I_{M_\alpha}(M_{\alpha+1}) = s(r_\alpha) \dot{+} n(r_\alpha)$ . But, on the other hand, the ideal  $s(r_\alpha)$  (as a ring) is special and the ideal  $n(r)$  — completely non-special ([4], Th. 9). Hence, by [4] (Th. 10), we have  $s(r_\alpha) \subseteq s(c_\alpha)$ ,  $n(r_\alpha) \subseteq n(c_\alpha)$  i.e.  $c_\alpha = s(c_\alpha) \dot{+} n(c_\alpha)$ .

Notice that, by Lemma 5,  $c_\alpha$  is strongly semi-simple as an ideal of the strongly semi-simple ring  $r_\alpha$ .

DEFINITION 4. Let  $r$  be a strongly semi-simple ring of the type  $\tau$ . Then the sequence of the decomposable and strongly semi-simple rings

$$(*) \quad c_0, c_1, \dots, c_\alpha, \dots, \quad (\alpha < \tau),$$

where  $c_\alpha = I(M_{\alpha+1})/I(M_\alpha)$ , will be called the *composition sequence* of  $r$ .

THEOREM 4. Let  $r$  be a strongly semi-simple ring of type  $\tau$  and let  $(*)$  be the composition sequence of  $r$ . Then

- (i) the rings  $c_\alpha$  ( $\alpha < \tau$ ) are decomposable and strongly semi-simple;
- (ii) for each  $\alpha < \tau$  there exists such an ascending sequence of special and strongly semi-simple rings  $s(c_\alpha) = s_{\alpha+1}(c_\alpha) \subset \dots \subset s_\xi(c_\alpha) \subset \dots \subset s_\tau(c_\alpha) = r'_\alpha = r_\alpha/n(r_\alpha)$ , ( $\alpha < \xi \leq \tau$ ), that for each  $\xi$  ( $\alpha < \xi < \tau$ ),  $n_\xi(c_\alpha)$  is an ideal of  $s_{\xi+1}(c_\alpha)$  and  $s_{\xi+1}(c_\alpha)/s_\xi(c_\alpha)$  is isomorphic with  $c_\xi$ ;
- (iii) for each  $\alpha < \tau$  there exists such an ascending sequence of completely non-special and strongly semi-simple rings  $n(c_\alpha) = n_{\alpha+1}(c_\alpha) \subset \dots \subset n_\xi(c_\alpha) \subset \dots \subset n_\tau(c_\alpha) = r''_\alpha = r_\alpha/s(r_\alpha)$ , ( $\alpha < \xi < \tau$ ), that, for each  $\xi$  ( $\alpha < \xi < \tau$ ),  $n_\xi(c_\alpha)$  is an ideal of  $n_{\xi+1}(c_\alpha)$  and  $n_{\xi+1}(c_\alpha)/n_\xi(c_\alpha)$  is isomorphic with  $c_\xi$ ;
- (iv) all rings  $s_\xi(c_\alpha)$  ( $\alpha < \xi \leq \tau$ ) are subdirectly imbedded into  $\sigma(\text{int } \bar{D}_{\alpha+1})$  in similar position as  $s(c_\alpha)$  and  $r'_\alpha$ . Furthermore all rings  $n_\xi(c_\alpha)$ , ( $\alpha < \xi \leq \tau$ ) are subdirectly imbedded into  $\sigma(M_\alpha - \bar{D}_{\alpha+1})$ .

Proof. The property (i) is equivalent to Theorem 3.

We put  $\mathfrak{f}_{\xi\alpha} = I(M_\xi)/I(M_\alpha)$ , ( $\alpha < \xi < \tau$ ). By Lemma 2, applied to the ring  $r_\alpha = r/I(M_\alpha)$ , we obtain  $\mathfrak{f}_{\xi\alpha} = I_{M_\alpha}(M_\xi)$ . In particular,  $\mathfrak{f}_{\alpha+1,\alpha} = I_{M_\alpha}(M_{\alpha+1}) = c_\alpha$  and  $\mathfrak{f}_{\tau\alpha} = I_{M_\alpha}(M_\tau) = I_{M_\alpha}(\emptyset) = r_\alpha$ . Notice that for a fixed  $\alpha$  the  $\mathfrak{f}_{\xi\alpha}$  ( $\alpha < \xi < \tau$ ) form an ascending sequence. Indeed, for  $\alpha < \xi < \xi' \leq \tau$  we have  $M_{\xi'} \supset M_\xi$  and  $I_{M_\alpha}(M_{\xi'}) \supset I_{M_\alpha}(M_\xi)$ . In particular  $\mathfrak{f}_{\xi\alpha} = I_{M_\alpha}(M_\xi) \supseteq I_{M_\alpha}(M_{\alpha+1}) = c_\alpha = s(c_\alpha) \dot{+} n(c_\alpha)$ . Thus we can put  $s_\xi(c_\alpha) = \mathfrak{f}_{\xi\alpha}/n(c_\alpha)$  and  $n_\xi(c_\alpha) = \mathfrak{f}_{\xi\alpha}/s(c_\alpha)$ , ( $\alpha < \xi \leq \tau$ ). In particular  $s_{\alpha+1}(c_\alpha) = s(c_\alpha) \dot{+} n(c_\alpha)/n(c_\alpha) \sim s(c_\alpha)$  and  $s_\tau(c_\alpha) = r_\alpha/n(c_\alpha) = r'_\alpha$ . Furthermore  $n_{\alpha+1}(c_\alpha) = s(c_\alpha) \dot{+} n(c_\alpha)/s(c_\alpha) \sim n(c_\alpha)$  and  $n_\tau(c_\alpha) = r_\alpha/s(c_\alpha) = r''_\alpha$ .

Let us notice that for  $\alpha < \xi < \tau$  we have

$$\begin{aligned} s_{\xi+1}(c_\alpha)/s_\xi(c_\alpha) &= \mathfrak{f}_{\xi+1,\alpha}/n(c_\alpha) / \mathfrak{f}_{\xi\alpha}/n(c_\alpha) \sim \mathfrak{f}_{\xi+1,\alpha}/\mathfrak{f}_{\xi\alpha} = \\ &= I_{M_\alpha}(M_{\xi+1})/I_{M_\alpha}(M_\xi) \sim I(M_{\xi+1})/I(M_\xi) = c_\xi. \end{aligned}$$

In an analogous way we can prove that  $n_{\xi+1}(c_\alpha)/n_\xi(c_\alpha) \sim c_\xi$ .

Finally let us notice that  $s(r_\alpha) = s(c_\alpha) \subseteq s_\xi(c_\alpha) \subseteq r'_\alpha$  and  $n(r_\alpha) = n(c_\alpha) \subseteq n_\xi(c_\alpha) \subseteq r''_\alpha$ , ( $\alpha < \xi \leq \tau$ ). Hence, by Lemma 3, Theorem 1, Corollary 2, and Lemma 5 we obtain that each ring  $s_\xi(c_\alpha)$  is special, strongly semi-simple and can be subdirectly imbedded into  $\sigma(\text{int } \bar{D}_{\alpha+1})$  in similar position as  $s(r_\alpha)$  and  $r'_\alpha$ . Furthermore, each ring  $n_\xi(r_\alpha)$  is completely non-special, strongly semi-simple and can be subdirectly imbedded into  $\sigma(M_\alpha - \bar{D}_{\alpha+1})$ .

In our next paper we shall prove that for each sequence (\*) of rings satisfying the conditions (i)—(iv) there exists such a strongly semi-simple ring of the type  $\tau$ , that (\*) is its composition sequence.

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# Mappings of Hilbert-Schmidt-Type. Their Application to Eigenfunction Expansions and Elliptic Boundary Problems

by

K. MAURIN

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Since the classic paper by Gelfand and Kostučenko [6] the abstract eigenfunction expansions became more and more general and the proofs simpler [2]—[7]. The use of nuclearity allowed the present author [8] (also C. Foiaç [5]) to obtain a form of the theorem on spectral expansions of abstract kernels and general eigenfunction expansions for (denumerable) sets of commuting observables, which was satisfying in several respects.

The author knows of special cases in which one can obtain sharper results by more elementary procedure. Moreover, nuclearity is a very strong property and is difficult to prove directly (cf., e.g., [9]): it is the strange element in the structure of Hilbert space.

In this paper linear mappings  $E \rightarrow F$  of separable Hilbert spaces  $E, F$  are explored, which reduce to classic Hilbert Schmidt (H.S.) operators in case of  $E = F$ . (By means of H.S.-operators Berezanskiĭ obtained very interesting results and his important paper [2] stimulated our investigations).

The use of H.S.-mappings allowed us to obtain a general theorem convenient for applications (Theorem 6); its proof is extremely simple. The theorems of Berezanskiĭ are immediate corollaries of it. Perhaps the most important result is the H.S.-property of the Sobolev—Rellich imbeddings

$$H^{m+k}(\Omega_N) \rightarrow H^k(\Omega_N) \text{ for } m > N/2, \quad k \geq 0.$$

From our Theorem 4 follows the nuclearity of the imbedding  $H_0^{2m+k}(\Omega_N) \rightarrow H_0^k(\Omega_N)$ ,  $m > N/2$ ,  $k \geq 0$ , proved by a more difficult method by L. Maurin and the author [9]. The same theorem allowed the present author to obtain a sharpened form of important theorems by L. Gårding, F. E. Browder, Yu. Berezanskiĭ, Gelfand and Kostučenko a.o. Also from the same theorem follows immediately a far reaching generalization of a recent theorem by Browder: a sufficiently high iteration  $R^j$ ,  $j > N/2$  of the resolvent  $R$  of a correctly posed elliptic boundary problem of order  $r$  is an integral operator of Hilbert-Schmidt type.

## Elementary properties of H. S.-mappings

DEFINITION. The linear mapping  $A: E \rightarrow F$  of the form

$$E \ni u \rightarrow Au \stackrel{df}{=} \sum_i (u, e_i)_E f_i = \sum_i (u, e_i)_E A e_i \in F,$$

where  $(e_i)_i^\infty$  is an orthonormal basis of (pre-) Hilbert space  $E$ ,  $f_i \in F$ ,  $\sum_i \|f_i\|_F^2 = \sum_i \|A e_i\|_F^2 < \infty$ , is called H.S.-mapping.

THEOREM 1. The H.S.-mapping is continuous; the H. S.-norm

$$|A| \stackrel{df}{=} (\sum \|A e_i\|_F^2)^{\frac{1}{2}} \geq \|A\|.$$

THEOREM 2. Let  $G, H$  be Hilbert spaces.  $B: G \rightarrow E$ ;  $C: F \rightarrow H$  are continuous,  $A: E \rightarrow F$  is of H.S.-type. Then  $B \circ A \circ C$  and the adjoint  $A^*: F \rightarrow E$  are of H.S.-type.

Let us recall the following

DEFINITION. The linear mapping  $A: E \rightarrow F$  is nuclear, if

$$E \ni u \rightarrow Au \stackrel{df}{=} \sum (u, g_i)_E h_i \in F, \quad g_i \in E, \quad h_i \in F,$$

where

$$\sum \|g_i\|_E \|h_i\|_F < \infty.$$

THEOREM 3. If  $A, B$  are H.S.-mappings, then  $B \circ A$  is nuclear.

## Applications to differential problems

Let us now recall the definition of important Hilbert spaces occurring in the theory of partial differential operators.

DEFINITION.

$$(u, v)_m \stackrel{df}{=} \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_0, \quad \text{where} \quad (u, v)_0 = \int_{\Omega_N} u \bar{v} dx,$$

$$D_{a_1} \dots D_{a_N}, \quad D_{a_k} = (\partial / \partial x_k)^{a_k}, \quad |\alpha| = a_1 + \dots + a_N.$$

$H_{\infty}^m(\Omega_N) = H^m$  (resp.  $H_0^m = H_0^m(\Omega_N)$ ) is the completion of  $C^\infty(\bar{\Omega}_N)$ , (resp.  $C_0^\infty(\Omega_N)$ )-functions with compact supports in  $\Omega_N$ ) in  $\|\cdot\|_m$ -norm,  $\Omega_N$  is bounded.

For any domain (bounded or not)  $\Omega_N$   $\mathcal{H}^m(\Omega_N) \stackrel{df}{=} 1$ . and  $H_0^m(\Omega^p)$ , where  $\Omega^p \nearrow \Omega_N$ ,  $\Omega^p$ -is bounded.

It is well known (Kondrasev—Sobolev) that the imbeddings  $H^{m+1}(\Omega_N) \rightarrow H^m(\Omega_N)$  are (completely) continuous; the following theorem affirms much more:

THEOREM 4 (fundamental). The imbeddings  $H^{m+k} \rightarrow H^k$  (resp.  $H_0^{m+k} \rightarrow H_0^k$ ) for bounded  $\Omega_N$  with strong cone-property, or any other Sobolev condition (resp. every bounded  $\Omega_N$ ) are H. S.-mappings for  $m > N/2$ ,  $k \geq 0$ .

From Theorems 3 and 4 one gets the following

COROLLARY 1. Imbeddings  $H^{2m+k}(\Omega_N) \rightarrow H^k(\Omega_N)$ ,  $m > N/2$ ,  $k \geq 0$  are nuclear.

COROLLARY 2. (a theorem by L. Maurin and K. Maurin [9]) *Imbeddings*  $H_0^{2m+k}(\Omega_N) \rightarrow H_0^k(\Omega_N)$  for  $m > N/2$ ,  $k \geq 0$  are nuclear.

We call the elliptic boundary problem of order  $r$ :

$$Au = f, \quad B_j u|_{\partial\Omega_N} = 0 \quad j = 1, \dots, p,$$

(where  $A \stackrel{df}{=} \sum a_\alpha(x) D^\alpha$  and  $B_j$  are differential (boundary) operators on the boundary  $\partial\Omega_N$  of  $\Omega_N$ ) correctly posed, if there are positive constants  $c_k$  such that

$$\|u\|_{r+k} \leq c_k \|Au\|_k, \quad k = 0, 1, 2, \dots$$

(cf. [1], [10]).

F. E. Browder proved in 1954 that the power  $R^j$  of the resolvent (in  $H^\circ(\Omega_N)$ ) of a self-adjoint extension of elliptic operator of order  $r$  is of Carleman type, if  $jr > N/2$ . Recently [3] Browder obtained by an involved calculation that for a self-adjoint Dirichlet boundary problem  $R^j$  is a H.S. integral operator, while from our fundamental theorem the same follows immediately and quite generally for every correctly posed elliptic problem.

THEOREM 5. *For every correctly posed elliptic problem:*

$$(*) \quad Au = f, \quad B_j u|_{\partial\Omega_N} = 0 \quad j = 1, 2, \dots, m$$

of order  $r$  the power  $R^j$  of the resolvent of  $(*)$  is an integral operator of Hilbert Schmidt-type for  $rj > N/2$  i.e.

$$(R^j f)(x) = \int_{\Omega_N} a(x, y) f(y) dy, \quad \text{where} \quad \int_{\Omega_N \times \Omega_N} |a(x, y)|^2 dx dy < \infty.$$

#### Applications to general eigenfunction expansions

The concept of H.S.-mapping allows us to formulate (and to prove very easily) the following eigenfunction theorem of Gelfand—Kostučenko-type. Let us recall the concept of an  $(A_\beta)$ -Fourier-transformation.

DEFINITION. Let  $(A_\beta)_{\beta \in B}$  be any (strongly) commuting set of self-adjoint operators in the separable Hilbert space  $H$ . Then there exists (essentially) a unique direct integral

$$\hat{H} = \int_A \hat{H}(\lambda) d\mu(\lambda) \quad \text{of Hilbert spaces} \quad \hat{H}(\lambda)$$

and a unitary mapping  $F: H \rightarrow \hat{H}$  — the so-called  $(A_\beta)$ -Fourier transformation — which diagonalises simultaneously all  $A_\beta$ .

$$H \ni \varphi \rightarrow F\varphi = \hat{\varphi} \in \hat{H}. \quad A \ni \lambda \rightarrow \hat{\varphi}_k(\cdot) \in C^1,$$

$$(\hat{A}\hat{\varphi})_k(\lambda) = \hat{A}_\beta(\lambda)\hat{\varphi}_k(\lambda), \quad k = 1, 2, \dots, \dim \hat{H}(\lambda),$$

where  $\hat{A}_\beta(\lambda) \in E^1$ .

THEOREM 6. If  $\Phi$  is an inductive limit of Hilbert spaces  $H_p \subset H$ ,  $\Phi = \text{l. ind. } H_p$ , where the canonic imbeddings  $i_p: H_p \rightarrow H$  are H.S., then the  $(A_\beta)$ -Fourier transformation of  $\varphi$  has the following form for  $\mu - \text{a.e. } \lambda \in \Lambda$ :

$$\hat{\varphi}_k(\lambda) = \langle \varphi, e_k(\lambda) \rangle \quad k = 1, 2, \dots, \dim \hat{H}(\lambda),$$

where  $e_k(\lambda) \in \Phi'$  (dual of  $\Phi$ ). The linear functionals  $e_k(\lambda)$  are simultaneous eigenfunctions of  $(A_\beta)_{\beta \in B}$ .

Immediately we obtain the following important corollaries:

COROLLARY 3. (Berezanskiĭ). If the operator  $B$  with dense domain  $D(B)$  has an inverse  $B^{-1}$  of H.S.-type, then putting  $H_1 = D(B)$  with  $(u, v)_B = (Bu, Bv) + (u, v)$  (scalar product in the graph of  $B$ ) we obtain the thesis of Theorem 6.

Other spectral theorems of Berezanskiĭ immediately follow from Theorem 6.

COROLLARY 4. The eigenelements of partial differential operators  $(A_\beta)$  on  $\Omega_N$  are distributions  $e_k(\lambda) \in \mathcal{H}^{-m}(\Omega_N) = \mathcal{H}^{m'}(\Omega_N)$   $m < N/2$  (their order does not exceed  $N/2$ ).

Proof. Put in Theorem 5:  $H_p = H^m(\Omega^p)$ ,  $H = L^2(\Omega_N)$ ,  $\Omega^p \nearrow \Omega_N$ .

Now let  $(u, v) = b(u, v)$  be a scalar product on  $C_0^\infty(\Omega_N)$  of order  $\leq r$ . This means that to any precompact subset  $K$  of  $\Omega_N$ , there is a constant  $c = c(K)$  such that

$$b(u, v) \leq c \|v\|_r \|u\|_r, \quad u, v \in C_0^\infty(\Omega_N).$$

Let  $H(b)$  be the Hilbert space obtained by completing  $C_0^\infty(\Omega_N)$  in the norm  $(b(u, u))^{1/2}$ . From Theorem 5 and the fundamental Theorem 4 we obtain the following sharpened form and generalisations of various theorems by Gårding, Berezanskiĭ a.o. ([2], [4]—[7]).

THEOREM 7. Let  $(A_\beta)_{\beta \in B}$  be a commuting set of self-adjoint operators in  $H(b)$ , then for almost all  $\lambda$  the components  $\hat{\varphi}_k(\lambda)$  of the  $(A_\beta)$ -Fourier transformation of  $\varphi$  are of the following form

$$q_k(\lambda) = \langle q, e_k(\lambda) \rangle, \quad k = 1, 2, \dots, \hat{H}(\lambda),$$

where  $e_k(\lambda) \in H'(b)$ , therefore the simultaneous eigenfunctions  $e_k(\lambda)$  are elements of  $H^{-(m+r)}(\Omega_N)$ , where  $m < N/2$  and  $r$  is the order of  $b(\cdot, \cdot)$ .

These results were reported on several occasions: the 2nd Hungarian Math. Congress at Budapest, Banach Memorial Conference at Jablonna (both in September, 1960) and the 5th Austrian Math. Congr. at Innsbruck October 1960. For complete proofs see a forthcoming paper in Math. Scand.

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# Sur la fixation et l'enfilage des ensembles compacts

par

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$E$  étant un espace métrique, un ensemble  $X \subset E$  sera dit *fixable dans  $E$*  (cf. [1]) lorsque, pour tout  $\varepsilon > 0$ , il existe dans  $E$  une somme finie  $G_\varepsilon = G_1 \cup G_2 \cup \dots \cup G_{k(\varepsilon)}$ , d'ensembles ouverts tels que  $\delta(G_i) < \varepsilon$  pour  $i = 1, \dots, k(\varepsilon)$ , que  $\bar{G}_i \cap \bar{G}_j \neq \emptyset$  pour  $i \neq j$  et que  $G_\varepsilon \cap C \neq \emptyset$  pour toute composante  $C$  de  $X$ . Cette définition équivaut trivialement à celle qu'elle devient en remplaçant les ensembles ouverts  $G_\varepsilon$  et  $G_i$  par les ensembles fermés  $F_\varepsilon$  et  $F_i$  respectivement.

Il existe dans l'espace euclidien  $\mathcal{L}^3$  des ensembles non fixables déjà parmi les  $X$  compacts formés d'une suite  $\{C_n\}$  de composantes de diamètre  $\delta(C_n) \geq 1$  (voir [1], Exemple I).

Knaster appelle dans son séminaire un ensemble  $X \subset E$  *enfilable dans  $E$*  lorsque  $E$  contient un arc  $L$  tel que  $L \cap C \neq \emptyset$  pour toute composante  $C$  de  $X$ . Par exemple, tout  $X$  compact de dimension 0 est enfilable dans  $\mathcal{L}^2$  d'après le théorème classique de Riesz et Denjoy ([2], p. 385) de même que dans tout continu  $E$  localement connexe sans points de séparation locale d'après une généralisation de Whyburn ([3], p. 57). Par contre, l'ensemble compact formé par le segment  $0 \leq x \leq 1$  de l'axe d'abscisses et par les points du plan  $((2k-1)/2^n, 1/2^n)$  où  $n = 1, 2, \dots$  et  $k = 1, \dots, 2^{n-1}$  n'est pas enfilable; il n'est pas fixable non plus.

Les deux notions sont en effet en relation étroite. La présente communication a pour but de signaler quelques résultats concernant cette relation pour les  $X$  compacts dans les espaces euclidiens  $\mathcal{L}^n$  où  $n > 1$ . En particulier, la possibilité d'y enfiler un tel ensemble entraîne toujours celle de l'y fixer, mais la réciproque n'est vraie que pour le plan. Il y a donc ici une différence essentielle entre  $\mathcal{L}^2$  et  $\mathcal{L}^n$  où  $n \geq 3$ . Cependant, l'existence de l'enfilage y équivaut, comme dans  $\mathcal{L}^2$ , à celle de la fixation au sens plus strict et que j'appelle *monotone*, à savoir lorsque, en outre, pour toute suite  $\{\varepsilon_n\}$  décroissante et convergente vers 0, les  $F_{\varepsilon_n}$  qui fixent  $X$  peuvent être choisis de manière qu'ils forment une suite descendante.

Je fais intervenir deux notions auxiliaires suivantes:

Tout ensemble  $R \subset X$  tel que  $R \cap C \neq \emptyset$  pour toute composante  $C$  de  $X$  sera dit un *réduit* de  $X$ ; en particulier, tout  $X$  est donc un réduit de lui-même et tout  $X$  de dimension 0 en est un réduit unique.

L'ensemble  $A$  de tous les points  $p$  de  $E$  tels que  $(p) = \lim_{n \rightarrow \infty} C_n$  pour une suite  $\{C_n\}$  de composantes de  $X$  s'appellera l'*adduit* de  $X$ ; ainsi défini,  $A$  est donc l'ensemble de tous les points de  $E$  qui sont des points-limites des suites de points appartenant à des composantes  $C_n$  de  $X$  telles que  $\delta(C_n)$  tend à 0 et de tous les points de  $X$  isolés. On voit aussitôt qu'un adduit est toujours fermé, donc compact pour  $X$  compact; on voit aussi qu'une fixation de  $X$  est nécessairement un recouvrement de son adduit.

On a d'abord le

THÉOREME 1. *Les trois propriétés suivantes sont équivalentes pour les  $X$  compacts dans  $\mathcal{E}^n$  où  $n > 1$ :*

- (1) *l'existence d'une fixation monotone de  $X$ ,*
- (2) *l'existence dans  $X$  d'un réduit  $R$  compact de dimension 0,*
- (3) *l'existence d'un enfilage de  $X$ .*

En effet, (1) entraîne (2). Les  $F_{\varepsilon_n}$  fermés qui fixent  $X$  formant par hypothèse une suite descendante, l'ensemble  $R = \bigcap_{n=1}^{\infty} F_{\varepsilon_n} \cap X$  est compact et ne contient aucun continu de diamètre positif, puisque les  $F_{\varepsilon_n}$  sont, par définition, de diamètre inférieur à lui à partir d'un  $n$  suffisamment élevé.  $R$  est donc ponctiforme — propriété qui, pour les ensembles compacts, coïncide avec leur dimension 0. Enfin,  $C$  étant une composante de  $X$ ,  $\{F_{\varepsilon_n} \cap C\}$  est une suite descendante d'ensembles compacts, d'où  $R \cap C = \bigcap_{n=1}^{\infty} F_{\varepsilon_n} \cap C \neq \emptyset$  en vertu du théorème de Cantor.  $R$  est donc un réduit de  $X$ .

(2) entraîne (3), car il suffit d'enfiler  $R$  — ce qui est possible dans  $\mathcal{E}^n$  en vertu du théorème précité de Whyburn — pour que  $X$  se trouve, par définition, enfilé et sur le même arc  $L$  que  $R$ .

Enfin, (3) entraîne (1), car la frontière sur un arc de tout vrai sous-ensemble compact est de dimension 0; telle est donc en particulier la frontière  $F$  de  $X \cap L$  sur  $L$ , ce qui entraîne aussitôt l'existence d'une fixation monotone de  $F$  (exprimée en d'autres termes dans le corollaire et la seconde remarque, p. 59, de [2]) dans un domaine fermé de  $\mathcal{E}^n$ , donc dans  $\mathcal{E}^n$ . Or, la frontière  $F$  d'un vrai sous-ensemble compact d'un continu ayant des points communs avec chacune de ses composantes (voir [2], p. 113, 3), toute fixation monotone de  $F$  en est à la fois une de  $X \cap L$ .

On a ensuite les trois théorèmes, dont les démonstrations, de même que les constructions des exemples qui s'y rattachent, seront contenues dans mon travail en préparation pour *Fundamenta Mathematicae*:

THÉOREME 2. *Si un  $X$  (compact ou non) est fixable dans  $\mathcal{E}^n$  où  $n \geq 1$ , son adduit est vide ou de dimension 0.*

THÉOREME 3. *Si toute composante  $C$  d'un  $X \subset \mathcal{E}^2$  compact est de diamètre  $\delta(C) \geq 1$ , il existe dans  $X$  un réduit compact de dimension 0.*

THÉOREME 4. *Si l'adduit d'un  $X \subset \mathcal{E}^2$  compact est vide ou de dimension 0, il existe dans  $X$  un réduit compact de dimension 0.*

La fixation monotone étant, par définition, un cas particulier de la fixation tout court, il résulte du Théorème 1 que, pour  $X$  compacts, l'existence d'un enfilage de  $X$  dans  $\mathcal{L}^n$  où  $n > 1$  y entraîne celle d'une fixation de  $X$ . Un exemple dans  $\mathcal{L}^3$  dû à Lelek montre que l'implication inverse n'est pas vraie. Théorèmes 2 et 4 impliquent toutefois que, pour  $X$  compacts situés dans  $\mathcal{L}^2$ , la possibilité d'y fixer  $X$  entraîne celle de l'y enfiler. En particulier, tout  $X$  compact situé dans  $\mathcal{L}^2$  et dont les composantes sont de diamètre au moins 1 y est, en vertu des Théorèmes 1 et 3, enfilable et fixable à la fois. Enfin, chacune des deux dernières propriétés est, pour  $X \subset \mathcal{L}^2$  compacts, équivalente, en vertu des Théorèmes 1, 2 et 4, à ce que l'adruit de  $X$  est vide ou de dimension 0.

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## Separable Theories

by

A. EHRENFEUCHT

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A theory  $T$  is separable if there exists a recursive set  $X$  of formulas, such that

- ( $\alpha$ ) if  $\Phi$  is a theorem of  $T$ , then  $\Phi \in X$ ,  
( $\beta$ ) if  $\neg \Phi$  is a theorem of  $T$ , then  $\Phi \notin X$ .

Undecidable sentences can be either in  $X$  or in the complement of  $X$ .

Obviously, any unseparable theory is essentially undecidable [1]. We shall show in par. 2 that the converse theorem is not true. In par. 1 we shall give a necessary and sufficient condition of separability.

**1. THEOREM 1.** *A theory  $T$  is separable if and only if there exists a recursively enumerable set  $R$  of formulas, such that*

- a) if  $\Phi \in R$ , then  $\Phi$  is consistent with  $T$ ,  
b) for any closed formula  $\Phi$  there exists such  $\Psi \in R$ , that either  
 $\vdash \Psi \supset \Phi$  or  $\vdash \Psi \supset \neg \Phi$ .

( $\vdash \theta$  means that  $\theta$  is provable in the Predicate Calculus (PC)) \*).

**Proof.** If  $T$  is separable, then we consider an arbitrary set  $X$  satisfying ( $\alpha$ ) and ( $\beta$ ). We put  $\Psi \in R$ , if

either  $\Psi \in X$ ,  
or  $\Psi$  and  $\neg \Psi$  are not in  $X$ .

It is easy to show, that  $R$  satisfies the conditions a) and b).

Now, let us assume that there exists a recursively enumerable set  $R$  satisfying a) and b).

Let  $\theta_1, \theta_2, \dots$  be a recursive sequence of all theorems of Predicate Calculus and let  $\Psi_1, \Psi_2, \dots$  be a recursive sequence of all formulas in  $R$ . We define the set  $X$  as follows:

$\Phi \in X$  if and only if there exists such  $n$ , that a certain  $\theta_j$  has the form  $\Psi_i \supset \Phi$  ( $i, j \leq n$ ), and no  $\theta_k$  ( $k < j$ ) has the form  $\Psi_l \supset \neg \Phi$  ( $l < n$ ).

By a) and b)  $X$  is recursive and satisfies conditions ( $\alpha$ ) and ( $\beta$ ).

\* We do not need to assume that  $T$  is axiomatizable.

**THEOREM 2.** *An axiomatizable theory  $T$  is unseparable if and only if for any recursive family  $\{T_i\}$  of axiomatizable consistent extensions of  $T$ , there is a closed formula  $\Phi$  undecidable in each  $T_i$ .*

**Proof.** The set  $R$  of theorems of all  $T_i$ ,  $i = 1, 2, \dots$ , is recursively enumerable. Therefore, if  $T$  is unseparable, then, by Theorem 1, there exists a sentence  $\Phi$  such that neither  $\Psi \supset \Phi$ , nor  $\Psi \supset \neg \Phi$  are provable in PC. Hence, neither  $\Phi$  nor  $\neg \Phi$  are theorems of any  $T_i$  (otherwise, we might put  $\Psi = \Phi$  or  $\Psi = \neg \Phi$ ).

Conversely, if  $T$  is separable, then the family of extensions  $T \cup \{\Psi\}$ , where  $\Psi \in X$ , or  $\Psi \notin X$  and  $\neg \Psi \notin X$ , is recursive, and each theory  $T \cup \{\Psi\}$  is consistent (by  $(\alpha)$  and  $(\beta)$ ). Evidently, for any  $\Phi$ , either the extension  $T \cup \Phi$ , or  $T \cup \{\neg \Phi\}$  is in the family.

**Remark.** Unseparability of Peano Arithmetic was proved in [2]. The existence of an undecidable sentence for any recursively enumerable family of extensions of Peano Arithmetic is proved in [3]. Theorem 2 establishes the relation between these two results.

2. Theorem 2 shows that there is a close analogy between the concepts of unseparability and of essential undecidability. Indeed, the last concept could be defined as follows: an axiomatizable theory is essentially undecidable, if and only if for any axiomatizable consistent extension  $T_1$  of  $T$ , there is a closed formula  $\Phi$  undecidable in  $T_1$ .

Now, we shall construct an essentially undecidable but separable theory  $V$ . Theory  $V$  will be formulated in the propositional calculus, more exactly, it will contain infinitely many propositional constants and formulas obtained from them by propositional functors. The problem of a strictly finitistic example, i.e., of a theory which is formulated in the first order Predicate Calculus with the finite number of constants (predicates, functors) is still open.

**Preliminary construction.** Let  $\Omega$  be a set of finite 0,1 sequences such that:

- 1) The empty sequence  $\varepsilon \in \Omega$ ,
- 2)  $\langle s_1 \dots s_k \rangle \in \Omega$  implies that  $\langle s_1 \dots s_k 0 \rangle \in \Omega$  if and only if  $\langle s_1 \dots s_k 1 \rangle \in \Omega$ ,
- 3) if  $\langle s_1 \dots s_k \rangle \in \Omega$ , then  $\langle s_1 \dots s_{k-1} \rangle \in \Omega$ ,
- 4)  $\Omega$  is infinite,
- 5)  $\Omega$  is recursive,
- 6) there is no such recursive zero-one sequence  $(t_i)$ ,  $i = 0, 1, 2, \dots$ , that  $\langle t_1 \dots t_k \rangle \in \Omega$  for all  $k$ ,
- 7) there is a recursively enumerable infinite set  $\Delta \subset \Omega$  such that, if  $\langle s_1 \dots s_k \rangle \in \Delta$ , then for any  $n > k$  there exists a sequence  $\langle s_1 \dots s_k r_{k+1} \dots r_n \rangle \in \Omega$ .

Speaking intuitively,  $\Omega$  is a binary recursive tree, which has no recursive and infinite branch 1)–6). Condition 7) establishes that there is a recursively enumerable set of vertices which are on infinite branches. The existence of a tree  $\Omega'$  satisfying conditions 1)–6) only is well known [4]. In order to define a tree satisfying conditions 1)–7), we consider the set  $\Omega$  of 0,1 sequences defined as follows:  $\langle s_1 \dots s_n \rangle \in \Omega$  if and only if either  $\langle s_1 \dots s_n \rangle \in \Omega'$  or there exists  $k < n$  such that  $\langle s_1 \dots s_k \rangle \in \Omega'$ ,  $\langle s_1 \dots s_k 0 \rangle \notin \Omega'$  and  $\langle s_{k+1} \dots s_n \rangle \in \Omega'$ .

Roughly speaking,  $\Omega$  is obtained from  $\Omega'$  by attaching to each vertex in which a branch of  $\Omega'$  terminates another example of  $\Omega'$ . (This procedure is performed only once). It is easy to prove that, if  $\Omega'$  satisfies 1)–6), then  $\Omega$  satisfies 1)–7).

Construction of theory  $V$ . Let  $p_\alpha$  be propositional constants, where index  $\alpha$  is an arbitrary finite 0,1 sequence. Formulas of  $V$  are all formulas of propositional calculus built with these constants and functors  $\neg, \vee, \wedge, \supset, \equiv$ .

Theory  $V$  is based on the following axiom schemas:

- 1°  $p_e$ ,
- 2°  $\neg p_\alpha$  for  $\alpha \in \Omega$ ,
- 3°  $p_\alpha \equiv (p_{\alpha 0} \vee p_{\alpha 1})$ ,
- 4°  $p_{\alpha 0} \supset \neg p_{\alpha 1}$ .

Theory  $V$  is essentially undecidable. Indeed, let  $V'$  be a complete extension of  $V$ . Then the set of indices  $\alpha$ , such that  $p_\alpha$  is a theorem of  $V'$ , is a branch of the tree  $\Omega$  (by 1°–4°). This set is not recursive (by 7), hence theory  $V'$  is undecidable.

Theory  $V$  is separable. Let us notice that:

- a) if  $\alpha \in \Delta$ , then  $p_\alpha$  is consistent with  $V$  (by 7)).
- b) if  $\alpha = \langle s_1 \dots s_k \rangle$ ,  $\beta = \langle t_1 \dots t_l \rangle$ ,  $k > l$ , then  $p_\alpha \supset p_\beta$  or  $p_\alpha \supset \neg p_\beta$  are theorems of  $V$  (by 1°–4°).
- c) if  $p_{\beta_1}, \dots, p_{\beta_s}$  are all propositional constants in a formula  $\Phi$ ,  $\beta_i = \langle b_{i_1} \dots b_{i_{k_i}} \rangle$ ,  $i = 1, \dots, s$ ,  $\alpha = \langle a_1 \dots a_k \rangle$ ,  $k > k_i$ ,  $i = 1, \dots, s$ , then  $p_\alpha \supset \Phi$  or  $p_\alpha \supset \neg \Phi$  are theorems of  $V$  (by b)).

Now let us define a set  $R$  of formulas, as follows:  $\Phi \in R$  if and only if there exist  $\alpha \in \Delta$  such that  $p_\alpha \supset \Phi$  is a theorem of  $V$ .

By a) and c), the set  $R$  satisfies the condition of Theorem 1, hence, by this theorem, theory  $V$  is separable.

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# A Note on Strong and Weak Convergence of Singular Integrals of Vector-valued Functions

by

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Let  $X$  be a Banach space with a norm  $\| \cdot \|$ , and  $X^*$  — the conjugate space of  $X$ . Let further  $x(s)$  denote a vector-valued function defined on an interval  $\langle a, b \rangle$  with values in  $X$ ,  $(B) \int_a^b x(s) ds$  resp.  $(P) \int_a^b x(s) ds$  — the Bochner resp. Pettis integral of  $x(s)$  over  $\langle a, b \rangle$ .

If  $x(s)$  is Bochner resp. Pettis integrable over  $\langle a, b \rangle$ , and

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} \|x(s) - x(t)\| ds = 0 \quad \text{resp.} \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} |x^* x(s) - x^* x(t)| ds = 0$$

for any  $x^* \in X^*$ , then we say that the point  $t \in \langle a, b \rangle$  is a strong resp. weak  $L$ -point of the function  $x(s)$ .

In general, we use the fundamental notions of functional analysis [1] and of the theory of vector-valued functions [2].

Many well-known theorems from the classical theory of singular integrals may be transferred to the case of vector-valued functions without essential changes. For example, the analogue of the classical theorem of Faddeev reads as follows:

Let  $K_n(s, t)$ ,  $(n = 1, 2, \dots)$  be a sequence of real-valued measurable functions defined on a square  $\langle a, b \rangle \times \langle a, b \rangle$ , and

(1) for every fixed  $t \in (a, b)$ ,  $K_n(\cdot, t) \in L \langle a, b \rangle$ ,  $(n = 1, 2, \dots)$ .

Then in order that

$$\Phi_n(x, t) = (B) \int_a^b x(s) K_n(s, t) ds \quad \text{resp.} \quad \Phi_n(x, t) = (P) \int_a^b x(s) K_n(s, t) ds$$

strongly resp. weakly converge to  $x(t)$  for any Bochner resp. Pettis integrable function  $x(s)$  having a strong resp. weak  $L$ -point at  $t \in (a, b)$ , it is necessary and sufficient that the following conditions be satisfied:

$$(2) \quad \lim_{n \rightarrow \infty} \int_a^\beta K_n(s, t) ds = 1 \quad \text{for any } \alpha, \beta \text{ such that } a \leq \alpha < t < \beta \leq b,$$

and

$$(2') \quad \limsup_{n \rightarrow \infty} \int_a^b \Psi_n(s, t) ds \leq \Omega(t) < \infty,$$

where

$$\Psi_n(s, t) = \begin{cases} \operatorname{ess\,sup}_{u \in \langle a, s \rangle} |K_n(u, t)|, & s \in \langle a, t \rangle, \\ \operatorname{ess\,sup}_{u \in \langle s, b \rangle} |K_n(u, t)|, & s \in \langle t, b \rangle, \end{cases}$$

and  $\Omega(t)$  is a finite, real-valued function of  $t \in (a, b)$ .

In this note we shall show that the weak convergence of singular integrals of vector-valued functions is in fact more general than the strong one.

In the sequel, we shall always suppose that:

I) the real-valued measurable functions  $K_n(s, t)$ , ( $n = 1, 2, \dots$ ) defined in  $\langle a, b \rangle \times (a, b)$  satisfy the conditions (1), (2) and

$$(3) \quad \lim_{n \rightarrow \infty} (B) \int_a^b x(s) K_n(s, t) ds = x(t)$$

for any Bochner integrable and bounded function  $x(s)$  at any of its points of strong continuity  $s = t$  (Evidently, (2) follows at once from (3));

II) there exists in the Banach space  $X$  a sequence of elements  $x_m$  ( $m = 1, 2, \dots$ ) which is weakly, but not strongly, convergent to an element  $x_0 \in X$ ;

III)  $R = \{t_k\}$  denotes a fixed enumerable set in  $(a, b)$ .

LEMMA 1. *There exists a sequence  $x_m \in X$  ( $m = 1, 2, \dots$ ) such that  $\|x_m\| = 1$  ( $m = 1, 2, \dots$ ),  $\lim_{m \rightarrow \infty} x^* x_m = 0$  for any  $x^* \in X^*$ . The proof is trivial.*

LEMMA 2. *For any bounded Bochner integrable function  $x(s)$  defined on  $\langle a, b \rangle$ , we have*

$$\lim_{n \rightarrow \infty} (B) \int_{\gamma}^{\delta} x(s) K_n(s, t) ds = 0, \quad \text{where } \langle \gamma, \delta \rangle \subset \langle a, b \rangle \text{ and } t \notin \langle \gamma, \delta \rangle.$$

Proof. Let  $x_0(s) = \begin{cases} x(s), & s \in \langle \gamma, \delta \rangle, \\ 0, & s \in \langle a, b \rangle - \langle \gamma, \delta \rangle. \end{cases}$

By the condition (3), we have for  $t \notin \langle \gamma, \delta \rangle$

$$\lim_{n \rightarrow \infty} (B) \int_{\gamma}^{\delta} x(s) K_n(s, t) ds = \lim_{n \rightarrow \infty} (B) \int_a^b x(s) K_n(s, t) ds = x_0(t) = 0.$$

LEMMA 3. *To each fixed positive integer  $k$ , there exists a sequence of positive integers  $\{n_l^{(k)}\}$  and two sequences of real numbers  $\{\alpha_l^{(k)}\}$ ,  $\{\beta_l^{(k)}\}$  such that*

$$n_1^{(k)} < n_2^{(k)} < \dots < n_l^{(k)} < \dots,$$

$$a = \alpha_0^{(k)} < \alpha_1^{(k)} < \dots < \alpha_l^{(k)} < \dots < t_k, \quad \alpha_l^{(k)} \nearrow_{l \rightarrow \infty} t_k,$$

$$t_k < \dots < \beta_l^{(k)} < \dots < \beta_1^{(k)} < \beta_0^{(k)} = b, \quad \beta_l^{(k)} \searrow_{l \rightarrow \infty} t_k,$$

and, further, the following inequalities hold:

$$\left| \int_{E_l^{(k)}} K_{n_l^{(k)}}(s, t_k) ds - 1 \right| < \frac{1}{4}, \quad (l = 1, 2, \dots),$$

$$\left\| (B) \int_{\langle a, b \rangle - E_l^{(k)}} x_k(s) K_{n_l^{(k)}}(s, t_k) ds \right\| < \frac{1}{2^k} \cdot \frac{2}{4}, \quad (l = 1, 2, \dots),$$

where

$$E_l^{(k)} = \{ \langle \alpha_{l-1}^{(k)}, \alpha_l^{(k)} \rangle \cup \langle \beta_l^{(k)}, \beta_{l-1}^{(k)} \rangle \},$$

$$x_k(s) = \begin{cases} 0, & s = t_k, \\ \frac{x_m}{2^k}, & s \in E_{2m-1}^{(k)}, \\ \frac{1}{2^k} \left[ \frac{\alpha_{2m}^{(k)} - s}{\alpha_{2m}^{(k)} - \alpha_{2m-1}^{(k)}} x_m + \frac{s - \alpha_{2m-1}^{(k)}}{\alpha_{2m}^{(k)} - \alpha_{2m-1}^{(k)}} x_{m+1} \right], & s \in \langle \alpha_{2m-1}^{(k)}, \alpha_{2m}^{(k)} \rangle, \\ \frac{1}{2^k} \left[ \frac{\beta_{2m}^{(k)} - s}{\beta_{2m}^{(k)} - \beta_{2m-1}^{(k)}} x_m + \frac{s - \beta_{2m-1}^{(k)}}{\beta_{2m}^{(k)} - \beta_{2m-1}^{(k)}} x_{m+1} \right], & s \in \langle \beta_{2m}^{(k)}, \beta_{2m-1}^{(k)} \rangle, \end{cases}$$

and  $\{x_m\}$  is the sequence discussed in Lemma 1.

Proof. Put  $\alpha_0^{(k)} = a$ ,  $\beta_0^{(k)} = b$  and let  $n_1^{(k)}$  be a positive integer such that

$$\left| \int_{\alpha_0^{(k)}}^{\beta_0^{(k)}} K_{n_1^{(k)}}(s, t_k) ds - 1 \right| < \frac{1}{5}.$$

Let us take  $\alpha_1^{(k)}$ ,  $\beta_1^{(k)}$  such that

$$\frac{\alpha_0^{(k)} + t_k}{2} < \alpha_1^{(k)} < t_k < \beta_1^{(k)} < \frac{t_k + \beta_0^{(k)}}{2},$$

$$\left| \int_{E_1^{(k)}} K_{n_1^{(k)}}(s, t_k) ds - 1 \right| < \frac{1}{4},$$

$$\int_{\langle a, b \rangle - E_1^{(k)}} |K_{n_1^{(k)}}(s, t_k)| ds < \frac{1}{4}.$$

Suppose that we have defined  $n_1^{(k)}, \dots, n_{l-1}^{(k)}$ ;  $\alpha_0^{(k)}, \dots, \alpha_{l-1}^{(k)}$ ;  $\beta_0^{(k)}, \dots, \beta_{l-1}^{(k)}$ ; then the function  $x_k(s)$  is defined in the set  $E_1^{(k)} \cup \dots \cup E_{l-1}^{(k)}$ . Let  $n_l^{(k)} > n_{l-1}^{(k)}$  be such that

$$\left\| (B) \int_{E_1^{(k)} \cup \dots \cup E_{l-1}^{(k)}} x_k(s) K_{n_l^{(k)}}(s, t_k) ds \right\| < \frac{1}{2^k} \cdot \frac{1}{4},$$

$$\left| \int_{\alpha_{l-1}^{(k)}}^{\beta_{l-1}^{(k)}} K_{n_l^{(k)}}(s, t_k) ds - 1 \right| < \frac{1}{5}.$$

Let us further take  $\alpha_l^{(k)}, \beta_l^{(k)}$  so that

$$\frac{\alpha_{l-1}^{(k)} + t_k}{2} < \alpha_l^{(k)} < t_k < \beta_l^{(k)} < \frac{t_k + \beta_{l-1}^{(k)}}{2}$$

and

$$\left| \int_{E_l^{(k)}} K_{n_l^{(k)}}(s, t_k) ds - 1 \right| < \frac{1}{4}, \quad \int_{\alpha_l^{(k)}}^{\beta_l^{(k)}} |K_{n_l^{(k)}}(s, t_k)| ds < \frac{1}{4},$$

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In this way, we defined  $\{n_l^{(k)}\}$ ,  $\{\alpha_l^{(k)}\}$  and  $\{\beta_l^{(k)}\}$  so that  $n_l^{(k)} \nearrow \infty$ ,  $\alpha_l^{(k)} \nearrow t_k$ ,  $\beta_l^{(k)} \searrow t_k$  as  $l \rightarrow \infty$ ; hence the function  $x_k(s)$  is defined everywhere in  $\langle a, b \rangle$  in such a manner that

$$\begin{aligned} & \left\| (B) \int_{\langle a, b \rangle - E_l^{(k)}} x_k(s) K_{n_l^{(k)}}(s, t_k) ds \right\| \leq \left\| (B) \int_{E_l^{(k)} \cup \dots \cup E_{l-1}^{(k)}} x_k(s) K_{n_l^{(k)}}(s, t_k) ds \right\| + \\ & + \left\| (B) \int_{\alpha_l^{(k)}}^{\beta_l^{(k)}} |K_{n_l^{(k)}}(s, t_k)| ds \right\| \leq \frac{1}{2^k} \cdot \frac{1}{4} + \frac{1}{2^k} \int_{\alpha_l^{(k)}}^{\beta_l^{(k)}} |K_{n_l^{(k)}}(s, t_k)| ds \leq \frac{1}{2^k} \left( \frac{1}{4} + \frac{1}{4} \right) = \frac{1}{2^k} \cdot \frac{2}{4}, \end{aligned}$$

and

$$\left| \int_{E_l^{(k)}} K_{n_l^{(k)}}(s, t_k) ds - 1 \right| < \frac{1}{4}.$$

LEMMA 4. Let  $x(s) = \sum_{v=1}^{\infty} x_v(s)$ , where  $x_v(s)$  ( $v = 1, 2, \dots$ ) are the functions defined in the preceding Lemma. Then  $x(s)$  is bounded, Bochner integrable over  $\langle a, b \rangle$  and

$$\limsup_{n \rightarrow \infty} \left\| (B) \int_a^b [x(s) - x(t_k)] K_n(s, t_k) ds \right\| \geq \frac{1}{2^{k+2}}.$$

**Proof.** From the uniform convergence of the series we see that  $x(s)$  is bounded and Bochner integrable. Furthermore, we have

$$\begin{aligned} & \left\| (B) \int_a^b [x(s) - x(t_k)] K_{n_{2m-1}}^{(k)}(s, t_k) ds \right\| \geq \left\| (B) \int_{E_{2m-1}^{(k)}} [x_k(s) - x_k(t_k)] K_{n_{2m-1}}^{(k)}(s, t_k) ds \right\| - \\ & \quad - \left\| (B) \int_{\langle a, b \rangle - E_{2m-1}^{(k)}} [x_k(s) - x_k(t_k)] K_{n_{2m-1}}^{(k)}(s, t_k) ds \right\| - \\ & \quad - \left\| (B) \int_a^b \left\{ \sum_{\substack{v=1 \\ v \neq k}}^{\infty} [x_v(s) - x_v(t_k)] \right\} K_{n_{2m-1}}^{(k)}(s, t_k) ds \right\| = I_1 - I_2 - I_3. \end{aligned}$$

By Lemma 3 we have

$$I_1 = \frac{\|x_m\|}{2^k} \left\| \int_{E_{2m-1}^{(k)}} K_{n_{2m-1}}^{(k)}(s, t_k) ds \right\| \geq \frac{1}{2^k} \cdot \frac{3}{4}, \quad I_2 < \frac{1}{2^k} \cdot \frac{2}{4}.$$

From the definition of  $x_k(s)$  we see that it is strongly continuous in  $\langle a, b \rangle$ , except for the point  $t_k$ , whence  $\sum_{\substack{v=1 \\ v \neq k}}^{\infty} x_v(s)$  is strongly continuous at the point  $t_k$ , and by the condition (3) we have  $\lim_{m \rightarrow \infty} I_3 = 0$ . Hence

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left\| (B) \int_a^b [x(s) - x(t_k)] K_n(s, t_k) ds \right\| \geq \\ & \geq \limsup_{m \rightarrow \infty} \left\| (B) \int_a^b [x(s) - x(t_k)] K_{n_{2m-1}}^{(k)}(s, t_k) ds \right\| \geq \frac{1}{2^k} \left( \frac{3}{4} - \frac{2}{4} \right) - \lim_{m \rightarrow \infty} I_3 = \frac{1}{2^{k+2}}. \end{aligned}$$

**LEMMA 5.** The function  $x(s)$  from the preceding lemma is everywhere weakly continuous in  $\langle a, b \rangle$ .

**Proof.** By definition  $x_k(s)$  is strongly continuous in  $\langle a, b \rangle$  except for the point  $t_k$ , whence it is weakly continuous in the same set. It remains only to show the weak continuity of  $x_k(s)$  at  $t_k$ .

For any  $x^* \in X^*$ , we have

$$|x^* x_k(s) - x^* x_k(t_k)| = |x^* x_k(s)| \leq \begin{cases} \frac{1}{2^k} |x^* x_m|, & s \in E_{2m-1}^{(k)}, \\ \frac{1}{2^k} (|x^* x_m| + |x^* x_{m+1}|), & s \in E_{2m}^{(k)}. \end{cases}$$

Since  $\lim_{m \rightarrow \infty} x^* x_m = 0$ , therefore  $\lim_{\delta \rightarrow t_k} |x^* x_k(s) - x^* x_k(t_k)| = 0$ .

**THEOREM.** *There exists a vector-valued function  $x(s)$  such that it is bounded and Bochner integrable over  $\langle a, b \rangle$ , the sequence of singular Bochner integrals  $\Phi_n(x, t) = (B) \int_a^b x(s) K_n(s, t) ds$  is strongly convergent to  $x(t)$  in  $(a, b) - R$  (but does not strongly converge at any point of  $R$ ) and is everywhere weakly convergent to  $x(t)$  in  $(a, b)$ .*

**Proof.** Take into consideration the function  $x(s)$  defined in Lemma 4. For any  $x^* \in X^*$ ,  $x^* x(s)$  is a continuous real-valued function in  $\langle a, b \rangle$ . Let  $\tilde{x} \neq 0$  be any fixed element from  $X$ , then the vector-valued function  $\tilde{x} x^* x(s)$  is strongly continuous in  $\langle a, b \rangle$ . By the condition (3) we have for each  $t \in (a, b)$

$$\tilde{x} \left\{ \lim_{n \rightarrow \infty} \int_a^b x^* [x(s)] K_n(s, t) ds \right\} = \lim_{n \rightarrow \infty} (B) \int_a^b \{ \tilde{x} x^* [x(s)] K_n(s, t) \} ds = \tilde{x} x^* [x(t)],$$

whence we have for any  $x^* \in X^*$

$$\lim_{n \rightarrow \infty} x^* \left[ (B) \int_a^b x(s) K_n(s, t) ds \right] = \lim_{n \rightarrow \infty} \int_a^b x^* [x(s)] K_n(s, t) ds = x^* x(t).$$

This means that  $\Phi_n(x, t)$  is everywhere weakly convergent to  $x(t)$  in  $(a, b)$ .

From the strong continuity we see that  $\Phi_n(x, t)$  is strongly convergent to  $x(t)$  in  $(a, b) - R$ .

From Lemma 4 and the condition (2) we see that  $\Phi_n(x, t)$  does not strongly converge to  $x(t_k)$  at  $t_k$ . But from the weak convergence it follows that if  $\Phi_n(x, t)$  is strongly convergent at  $t_k$ , it must be strongly convergent to  $x(t_k)$ , whence  $\Phi_n(x, t)$  does not strongly converge at  $t_k$ .

**COROLLARY 1.** *The function  $x(s)$  defined in Lemma 4 has the following properties: the set of its strong  $L$ -points is  $\langle a, b \rangle - R$  and the set of its weak  $L$ -points is  $\langle a, b \rangle$ .*

**Proof.** Every point of strong resp. weak continuity of  $x(s)$  is at the same time a strong resp. weak  $L$ -point of it, and every strong  $L$ -point is also a weak one. So it is enough to show that  $R$  does not contain any strong  $L$ -point of  $x(s)$ .

Let us take any  $K_n(s, t)$  ( $n = 1, 2, \dots$ ) fulfilling the conditions (1), (2) (2'), then by the analogue of the theorem of Faddeev we see that the condition (3) is also fulfilled. Since in this case  $\Phi_n(x, t)$  in the preceding theorem is not strongly convergent at any point of  $R$ , therefore once more by the analogue of the theorem of Faddeev we see that  $x(s)$  has no strong  $L$ -point in  $R$ .

**COROLLARY 2.** *There exists a bounded Bochner integrable function  $x(s)$  on  $\langle a, b \rangle$  such that its indefinite Bochner integral is not strongly differentiable in a given enumerable set  $R \subset (a, b)$ , but is strongly differentiable in  $\langle a, b \rangle - R$  with the derivative  $x(s)$  and everywhere weakly differentiable with the derivative  $x(s)$ .*

Proof. Let  $K_n(s, t) = \begin{cases} n, & s \in \left\langle t, t + \frac{1}{n} \right\rangle \cap \langle a, b \rangle, \\ 0, & s \in \langle a, b \rangle - \left\langle t, t + \frac{1}{n} \right\rangle. \end{cases}$

Evidently,  $K_n(s, t)$  ( $n = 1, 2, \dots$ ) satisfy the conditions (1), (2) and (3). Let us consider the function  $x(s)$  defined in Lemma 4; the sequence

$$\Phi_n(x, t) = (B) \int_a^b x(s) K_n(s, t) ds = \frac{1}{1/n} (B) \int_t^{t+1/n} x(s) ds$$

is not strongly convergent in  $R$ , which means that the indefinite integral  $(B) \int_a^t x(s) ds$  is not strongly differentiable in  $R$ . The other parts of this corollary may be easily obtained by the strong continuity of  $x(s)$  in  $\langle a, b \rangle - R$  and the weak continuity of the same function in  $\langle a, b \rangle$ .

I would like to express my warmest thanks to professor W. Orlicz for his suggestions and encouragement in the course of preparation of this note.

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О. М. ФОМЕНКО

# О СУММЕ ЗНАЧЕНИЙ ХАРАКТЕРА

Представлено В. СЕРПИНСКИМ 26 ноября 1960 г.

Обозначения.  $c$  — положительное постоянное число;  $\varepsilon$  — произвольно малое положительное постоянное число;  $q$  — простое,  $q \geq c_0$ , где  $c_0$  — достаточно большое, превосходящее 2;  $\chi(a)$  — характер по модулю  $q$ , отличный от главного;  $k$  — постоянное целое, не делящееся на  $q$ ,  $0 < |k| < c_0$ ;  $p$  — переменное, пробегающее простые числа;  $l$  — целое положительное число;  $w$  — переменное, пробегающее произведения состоящие из  $l$  различных простых сомножителей; обозначение  $A \ll B$  при  $B > 0$  показывает, что  $|A| B^{-1}$  не превосходит положительного постоянного числа;  $\pi^{(l)}(n) = \sum_{w \leq n} 1$ .

В работе [1] И. М. Виноградов получил оценку

$$S_1 \ll N^{1+\varepsilon} \left( \frac{q^{3/4}}{N} \right)^{1/4},$$

справедливую при условии

$$q^{3/4} \ll N \ll q^{5/4},$$

которую в [2] он заменяет более точной оценкой

$$S_1 \ll N^{1+\varepsilon} \left( \left( \frac{q^{3/4}}{N} \right)^{1/3} + N^{-0,1} \right).$$

Оценки работ [1] и [2] нетривиальны при  $N > q^{0,75+\varepsilon}$ , что является большим достижением, поскольку расширенная гипотеза Римана для всех рядов  $L(s, \chi)$  с характерами (mod  $q$ ) непосредственно приводит лишь в оценке

$$S_1 \ll \frac{N}{\varphi(q)} + N \ln^2 N \sqrt{\frac{q}{N}},$$

которая нетривиальна при  $N > q \ln^4 q$ .

В настоящей заметке мы обобщаем результаты работ [1], [2] наводит лишь к оценке

$$S_l = \sum_{w \leq N} \chi(w+k).$$

ТЕОРЕМА 1. При условии  $q^{3/4} \ll N \ll q^{5/4}$  имеем

$$S_l \ll N^{1+\varepsilon} \left( \left( \frac{q^{3/4}}{N} \right)^{1/3} + N^{-0,1} \right).$$

ТЕОРЕМА 2. Пусть  $n$  — делитель числа  $q-1$  при условии  $1 < n < q-1$  и пусть  $s$  — одно из чисел  $0, 1, \dots, n-1$ . Тогда для числа  $T_S^{(l)}$  чисел вида  $w+k$  с условиями  $w \leq N$ ,  $\text{ind}(w+k) \equiv s \pmod{n}$  имеем неравенство

$$T_S^{(l)} - \frac{1}{n} \pi^{(l)}(N) \ll N^{1+\varepsilon} \left( \left( \frac{q^{3/4}}{N} \right)^{1/3} + N^{-0,1} \right).$$

Доказательство Теоремы 1 является развитием доказательства Теоремы 1 работы [2] при помощи идеи работы [3], причем наряду с леммами 1, 2, 3 работы [2] применяется некоторое обобщение Леммы 4 этой работы, а также следующая лемма

ЛЕММА. Пусть  $q$  — простое, большее 2,  $(k, q) = 1$ ;  $q^{0,5} \leq U \leq N$ ,  $0 < c < 0,5$ ,  $c \cdot U < U_0 \leq 0,5 U$ . Пусть  $x, y, t$  пробегает не делящиеся на  $q$  целые положительные числа, принадлежащие трем возрастающим последовательностям. Пусть

$$S = \sum_x \sum_y \sum_t \chi(xyt+k),$$

где суммирование распространяется на область

$$U - U_0 < x \leq U, \quad xyt \leq N, \quad (x, y) = 1.$$

Тогда

$$S \ll Nq^{\varepsilon} \sqrt{\frac{q^{0,5}}{U} + \frac{U}{N} + \frac{1}{q}}.$$

Теорема 2 выводится из Теоремы 1 способом, изложенным в работе [1]. Полное доказательство Теоремы 1 предназначено к печати в журнале Acta Arithmetica.

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O. M. FOMENKO, ON THE SUM OF THE VALUES OF A CHARACTER

The author proves the following Theorem:

Let  $\varepsilon$  be an arbitrary small positive number. Let  $q$  be a prime number,  $q \geq c_0$ , where  $c_0$  is sufficiently large.  $l$  will denote a positive integer;  $w$  runs over all products of  $l$  different prime factors. Let  $k$  be a fixed integer,  $(k, q) = 1$ ,  $0 < |k| < c_0$ . The symbol  $\chi(a)$  will denote the character with respect to the modulus  $q$  which is different from the main character. Then we have

$$\sum_{w \leq N} \chi(w+k) \ll N^{1+\varepsilon} \left( \left( \frac{q^{3/4}}{N} \right)^{1/3} + N^{-0.1} \right),$$

where  $q^{3/4} \ll N \ll q^{5/4}$  (for  $B > 0$  the symbol  $A \ll B$  will denote that  $|A|/B \leq \text{const.}$ ).



## Conservation Laws in General Relativity

by

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*Presented by L. INFELD on October 12, 1960*

### Energy-momentum

The trouble with the conservation laws in General Relativity\*) is that, owing to the covariant form of the continuity equations

$$(1) \quad T^{\mu\nu}_{;\nu} = 0,$$

where  $T^{\mu\nu}$  is the symmetrical energy-momentum tensor, their left hand sides are not of a form suitable for an application of the Gauss-Ostrogradzki theorem. However, as is well known, there is no difficulty with the conservation and localization of the electric charge in the Riemannian geometry, in spite of the fact that we have to do also in this case with a covariant continuity equation  $j^\mu_{;\mu} = 0$ . The absence of difficulty with the charge and considerable difficulty with the energy-momentum are due to the fact that covariant derivatives have a different meaning for each case of tensors of different ranks. The case of tensors of the first rank is, so to say, natural for deriving conservation laws from covariant continuity equations

$$(2) \quad V^\nu_{;\nu} = 0.$$

Indeed, by integrating (2) over an arbitrary four-dimensional volume we get an expression to which the Gauss theorem is directly applicable

$$(3) \quad 0 = \int_{\Omega} (\partial_\mu V^\mu + \Gamma^\mu_{\mu\nu} V^\nu) \sqrt{-g} dx = \int_{\Omega} \partial_\mu (V^\mu \sqrt{-g}) dx = \\ = \sum_{(\mu, \nu, \rho, \sigma)} \oint_S V^\mu \sqrt{-g} dS^{\nu\rho\sigma},$$

where the summation is extended over all even permutations of the indices. Introducing the element  $dS_\mu$  dual to  $dS^{\nu\rho\sigma}$

$$(4) \quad dS_\mu = \sqrt{-g} dS^{\nu\rho\sigma}$$

\*) Of the numerous papers on the subject we quote only a few that seem to have some similarity with our line of approach [1], [2].

and splitting the closed hypersurface  $S$  into two space-like hypersurfaces  $\Sigma_1$  and  $\Sigma_2$  we get

$$(5) \quad \int_{\Sigma_1} V^\mu dS_\mu = \int_{\Sigma_2} V^\mu dS_\mu,$$

whence the quantity

$$(6) \quad V = \int_{\Sigma} V^\mu dS_\mu = \int_{\Sigma} V^n dS$$

is independent of the choice of the space-like hypersurface, i.e. is conserved. The normal derivative  $V^n$  can be interpreted as the density of  $V$  at the points of  $\Sigma$ .

In the above derivation the form of the covariant derivative of the vector was taken advantage of explicitly. In the case of tensors of higher ranks, where the covariant derivatives involve further terms with Christoffel symbols, the analogous derivation obviously breaks down and it is impossible to infer the conservation laws in the above described way.

The above discussion not only shows wherefrom technical difficulties arise but, at the same time, suggests the way out of the difficulties: the tensor of energy-momentum is to be replaced by a suitable set of vectors. This end can be achieved by using the tetrad technique.

Let us introduce a tetrad field  $e_a^\mu$  (where Latin indices from the beginning of the alphabet, or numeral indices in parenthesis, denote the legs of the four-leg, i.e. of the tetrad) normalized as follows

$$(7) \quad e_{(1)\mu}^\mu e_{(1)\mu} = e_{(2)\mu}^\mu e_{(2)\mu} = e_{(3)\mu}^\mu e_{(3)\mu} = -e_{(4)\mu}^\mu e_{(4)\mu} = 1,$$

but not necessarily orthogonal one to the other.

By means of projections upon the legs of the tetrad we can replace any tensor of higher rank by an equivalent set of vectors. In particular, if  $W^{\mu\nu}$  denotes the energy-momentum tensor of the whole system including substantialistic matter and gravitation, we may form four fourvectors  $W_a^\nu$

$$(8) \quad W_a^\mu = e_a^\nu W_\nu^\mu,$$

which may be called "the energy-momentum currents", and look for a possibility of satisfying the following continuity equations

$$(9) \quad W_{a;\nu}^\nu = 0.$$

If we succeed in satisfying (9), then, by repeating exactly the derivation (2)–(6), we shall possess strictly conserved and localizable quantities

$$(10) \quad W_a = \int_{\Sigma} W_a^\mu dS_\mu$$

to be interpreted as the total energy-momentum components with respect to the given tetrad field  $e_a^\mu$ . The normal components  $W_a^n$  will be interpretable as the corresponding densities.

Thus, in order to guarantee conservation laws and localization of energy and momentum in General Relativity, we have to construct the "currents" satisfying the continuity equations (9). The Eqs. (9) can be used for determining the unknown tensor  $W^{\mu\nu}$ . But, as there are ten unknown functions  $W^{\mu\nu}$ , the four equations (9) can be satisfied in several ways and the determination of  $W^{\mu\nu}$  will not be unique. In order to determine  $W^{\mu\nu}$  uniquely from (9) some more information about the structure of  $W^{\mu\nu}$  is needed. It seems plausible to postulate the following general form of the symmetrical energy-momentum tensor

$$(11) \quad W^{\mu\nu} = Kg^{\mu\nu} + LR^{\mu\nu} + \partial^\mu M \partial^\nu N + \partial^\nu M \partial^\mu M,$$

where  $K, L, M, N$  are four unknown scalar functions to be determined from the four equations (9). Thus, we have at our disposal four differential equations of the first order (9) to determine four unknown functions.

These equations surely possess solutions if the metric tensor is sufficiently regular. For reasons of correspondence we have to look for a special solution of (9) which goes over into the energy-momentum-stress tensor of Special Relativity

$$(12) \quad W^{\mu\nu} \rightarrow T^{\mu\nu}$$

in the limit of the gravitational coupling constant tending to zero. Thus, we have to choose a special solution of (9) instead of solving in general the initial value problem for  $(\hat{\mathfrak{G}})$ . This seems satisfactory since the initial value problem is already settled in connection with Einstein's equations so that it would be an excessive arbitrariness, if we were to prescribe the initial values of  $W^{\mu\nu}$  independently of those of  $g^{\mu\nu}$ .

Splitting  $W^{\mu\nu}$  into two parts

$$(13) \quad W^{\mu\nu} = G^{\mu\nu} + U^{\mu\nu}, \quad \text{where} \quad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R,$$

the tensor  $G^{\mu\nu}$  can be interpreted, as usual, as the energy-momentum-stress tensor of the substantialistic matter, whereas the additional part  $U^{\mu\nu}$  which is necessary to secure the local energy-momentum balance can be interpreted as the energy-momentum-stress tensor of the gravitational field itself.

#### The choice of the tetrad field

The interpretation of  $W_a$  given by (10) as the total energy-momentum of the system is still arbitrary so long as the tetrad field is chosen arbitrarily.

In the limiting case of a flat space there exists a privileged class of tetrad fields, i.e. constant orthonormal fields. Expressed in terms of general curvilinear co-ordinates the requirement of going over into a constant and orthonormal field is

$$(14) \quad e^\mu_{a;\nu} \rightarrow 0, \quad e^\mu_a e_{b\mu} \rightarrow \eta_{ab}.$$

As is well known, such tetrad fields do not exist in the general Riemannian space. But it is intuitively clear that also in the case of a Riemannian space there should exist a class of tetrad fields corresponding closely to the class of constant orthonormal

tetrad fields. We shall try to define such a class by saving as much as possible from the constancy and orthonormality. Thus, if not all covariant derivatives can vanish simultaneously then, at least, some linear combinations of them can vanish everywhere. In particular, we may assume source-free tetrad fields  $e_a^\mu$

$$(15) \quad e_{a;\mu}^\mu = 0$$

being versors of curl-free tetrad fields  $h_a^\mu$

$$(16) \quad h_{a\mu;\nu} - h_{a\nu;\mu} = 0.$$

From (16) it follows that the tetrads  $h_a^\mu$  are derivable from potentials

$$(17) \quad h_{a\mu} = \partial_\mu \varphi_a.$$

The equations

$$(18) \quad \varphi_a = p_a$$

describe one-parametric families of extremal hypersurfaces. The conditions (15)—(17) determine the tetrad field uniquely, if we assume (improper) boundary conditions: the tetrads should go over into constant orthonormal tetrads (14), for points  $P$  tending to infinity in space-like directions, where the space can be assumed to be empty.

The tetrad fields defined by (15)—(17) with the boundary conditions (14) can be regarded as a natural generalization of the concept of the constant, orthonormal tetrad fields. The vector  $e_a^\mu$  at a point  $P$  can be regarded as “quasi-parallel” to the corresponding vector  $e_a^\mu$  at another point  $Q$ . Thus, the vector fields given by (15), (16) and (17) with the boundary conditions (14) define a sort of “fern-parallellism”.

The equations

$$(19) \quad \varphi_a = \text{const},$$

where  $\varphi_a$  are the potentials introduced by (17), define a co-ordinate system that can be regarded as a natural generalization of the conception of the inertial, Cartesian frame of reference and may be called “quasi-inertial”.

Changing the potentials by constant values corresponds to a translation (of the point  $x^\mu = 0$ ), while a Lorentz transformation of the boundary values (a rotation of the tetrads at space-like infinity) means a transformation between two quasi-inertial co-ordinate systems.

#### Angular momentum

In order to construct angular momentum in General Relativity we need a generalization of the concept of the radius vector

$$(20) \quad r_a = \int_{P_a}^P e_{a\mu} \frac{dx^\mu}{ds} ds.$$

In consequence of (18) this integral is independent of the path and can be written in the form

$$(21) \quad r_a = \int_{P_0}^P d\varphi_a = \varphi_a(P) - \varphi_a(P_0),$$

whence

$$(22) \quad \partial_\mu r_a = e_{a\mu}.$$

Now, we can introduce the angular momentum "currents"

$$(23) \quad M_{ab}^\mu = r_a W_b^\mu - r_b W_a^\mu$$

satisfying the continuity equations

$$(24) \quad M_{ab;\mu}^\mu = 0$$

in consequence of (9), (22), and the symmetry of  $W_{ab}$ . In this way also angular momentum with respect to an arbitrary point  $P_0$  is a strictly conserved and localizable quantity within the framework of General Relativity.

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## Estimation of the Influence of Foreign Atoms in the Core of Conjugated Bonds on the Spectrum and the Charge Distribution in the Molecule

by

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*Presented by A. JABŁOŃSKI on October 20, 1960*

The problem of estimation of the influence of foreign atoms or groups on a molecule has been discussed by various authors. Coulson and Longuet-Higgins [1] in a series of papers developed the perturbation theory based on the LCAO MO method. The method of estimation of the influence of perturbing atoms and groups on the distribution of electronic charge in the molecule, bond orders values and other molecular properties, have been given by these authors. The same problems have been also discussed in many other papers (see e.g. [2]—[5]). A theory of the perturbation of a system formed by two conjugated molecules or groups linked by hydrogen bridge has been given by Dewar [6]. A new mathematical perturbation method, called the method of perturbed secular determinant, has been proposed by a group of Japanese investigators Fukui et al. [7], who also gave formulae enabling computation of orbital energies of electrons, charge density distribution, and bond orders.

Molecules with systems of conjugated bonds may be easily investigated on the basis of the FEMO method. It seems possible to take into consideration the perturbing influence of foreign atoms also on the ground of this method by introducing an appropriate interaction potential of  $\pi$ -electrons with the core. In this work the calculations were carried out for pyridine and pyrimidine. As compared with the methods mentioned above, these calculations are much simpler. It seems therefore possible to extend them to much more complex systems.

A molecule of pyridine is represented in Fig. 1a. It differs from the benzene molecule (Fig. 1b) in that the CH group is replaced by an N atom. This molecule can be "constructed" by perturbing a benzene molecule by way of "pressing" a hydrogen atom into the neighbouring carbon atom. Thus, the number of  $\pi$ -electrons in the system does not change. The adopted form of potential along the core of a molecule is presented in Fig. 2. For the sake of simplicity a rectangular box of potential of the width  $2a = 1.36 \text{ \AA}$ , equal to the C—N bond length in pyridine,

has been assumed. Such a form of potential is connected with the fact, that nitrogen exhibits stronger electronegativity than carbon. Using the continuity conditions,

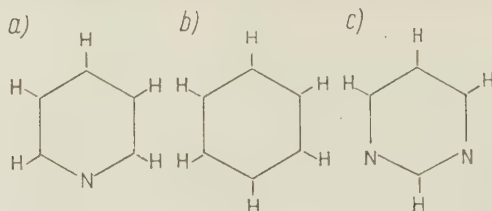


Fig. 1. Constitutional formulas of: a) pyridine, b) benzene, c) pyrimidine

which must be satisfied by a wave function and its derivatives, we obtain transcendental equations, which give the energy levels of a molecule; we have namely for  $-V_0 \leq E \leq 0$ .

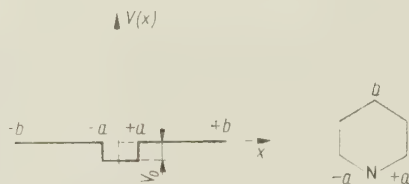


Fig. 2. Form of potential along the core of the molecule

$$k_2 \operatorname{tg} k_2 a - k_1 \operatorname{th} k_1 (b - a) = 0 \text{ for orbitals of species } b_2, *$$

$$k_2 \operatorname{ctg} k_2 a + k_1 \operatorname{cth} k_1 (b - a) = 0 \text{ for orbitals of species } a_2,$$

where

$$k_1^2 = -\frac{2m}{\hbar^2} E \text{ and } k_2^2 = \frac{2m}{\hbar^2} (V_0 + E);$$

and for  $E \geq 0$  we have

$$k_2 \operatorname{tg} k_2 a + k_1 \operatorname{tg} k_1 (b - a) = 0 \text{ for orbitals of species } b_2,$$

$$k_2 \operatorname{ctg} k_2 a + k_1 \operatorname{ctg} k_1 (b - a) = 0 \text{ for orbitals of species } a_2,$$

where

$$k_1^2 = \frac{2m}{\hbar^2} E \text{ and } k_2^2 = \frac{2m}{\hbar^2} (V_0 + E).$$

It is assumed that the C-C bond length equals 1.39 Å, and the C-N — 1.36 Å. The best agreement of the longest wave transition in pyridine with experiment is obtained for  $V_0 = 4.82$  eV. It seems reasonable to assume the same depth in each of the two potential boxes in pyrimidine (Fig. 1c). The calculations in this case are analogous.

\*) Pyridine belongs to the symmetry group  $C_{2v}$  [8].

The knowledge of the wave functions enables also to compute the oscillator strength of the electronic transition

$$f_{ik} = 2 \frac{8\pi^2 m}{3h} \nu_{ik} |\tilde{r}_{ik}|^2,$$

where  $\nu_{ik}$  is the frequency of transition, and  $\tilde{r}_{ik}$  the corresponding matrix element

$$\tilde{r}_{ik} = \int \Phi_i^* \tilde{r} \Phi_k dr.$$

The results of the calculations are given in the Table.

TABLE  
Spectra of pyridine and pyrimidine (first absorption band)

|            | Transition            | Calculated<br>$V_0 = 0$           |                     | Calculated<br>$V_0 = 4.82$ eV     |       | Calculated by<br><i>semi-empirical</i><br>method [9] |       | Experimental                      |                      |
|------------|-----------------------|-----------------------------------|---------------------|-----------------------------------|-------|--|-------|-----------------------------------|----------------------|
|            |                       | $\tilde{\nu}$ [cm <sup>-1</sup> ] | $f$                 | $\tilde{\nu}$ [cm <sup>-1</sup> ] | $f$   | $\tilde{\nu}$ [cm <sup>-1</sup> ]                    | $f$   | $\tilde{\nu}$ [cm <sup>-1</sup> ] | $f$                  |
| pyridine   | $A_1 \rightarrow B_1$ | 53200                             | 0.168 <sup>*)</sup> | 42800                             | 0.072 | 39500  | 0.047 | 39900 [10]                        | 0.041 <sup>xx)</sup> |
| pyrimidine | $A_1 \rightarrow B_1$ | —                                 | —                   | 44700                             | 0.042 | 40700  | 0.038 | 41500 [10]                        | —                    |

<sup>\*)</sup> The result of calculation of  $f$  is extremely flexible on taking the accurate value for the whole length of the ring. Considering for  $V_0 = 0$  the pyridine as a benzene ring, i.e. taking the length of each bond equal 1.39 Å, one obtains  $f = 2$ . It means that only the effect of contracting the free path of electrons corrects significantly the value obtained for oscillator strength. Proper agreement with experiment is, however, obtained by taking into account the existence of a potential box.

<sup>xx)</sup> Experimental data presented by H. P. Stephenson at Symposium on Molecular Structure and Spectroscopy, Ohio State University, June 9, 1952.

The first column contains the symmetry of states between which the transition takes place, the next two — appropriate wave numbers and oscillator strength computed on the basis of the FEMO method for  $V_0 = 0$ . The further columns contain analogous values computed for  $V_0 = 4.82$  eV, the result calculated by the *semi-empirical* method of Pariser and Parr [9], and experimental results. From the Table it is evident that, by taking into consideration the potential which is responsible for perturbation caused by the nitrogen atom, results in agreement with experimental ones as well as with those of Pariser and Parr will be obtained. Thus, we may infer that orbitals obtained for  $V_0 = 4.82$  eV describe fairly well the motion of  $\pi$ -electrons in a perturbed system.

These functions may be used for calculation of distribution of electronic charge and bond orders in a molecule. The "atom population" may be taken as value of the charge in the environment of an atom  $P$ . This value may be defined as follows [11]:

$$a(P) = \sum_n g_n \int_Q^R |\Phi_n|^2 dx,$$

where  $Q$  and  $R$  are co-ordinates of centres of two neighbouring bonds near atom  $P$ , and  $g_n = 0, 1, 2$  is the number of electrons occupying the orbital  $\Phi_n$ . The mobile bonds order  $p_{rs}$  between neighbouring atoms  $r$  and  $s$ , defined by Coulson and

Longuet-Higgins in the LCAO theory, in FEMO method ought to be defined as

$$p_{rs} = D_{rs} \sum_n g_n \Phi_n(r) \Phi_n(s),$$

where  $\Phi_n(r)$  and  $\Phi_n(s)$  are the values of  $\Phi_n$  for co-ordinates  $r$  and  $s$ , and  $D_{rs}$  is the distance between these atoms. The results of calculations for pyridine are given in Figs. 3 and 5. As can be seen, the character of the charge distribution obtained

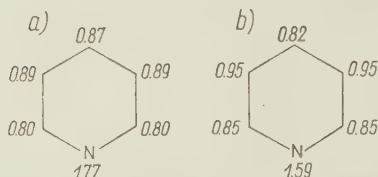


Fig. 3. Distribution of electronic charge in pyridine: a) present calculation, b) calculated by LCAO method [12]

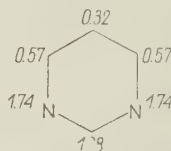


Fig. 4. Distribution of electronic charge in pyrimidine (present calculation)

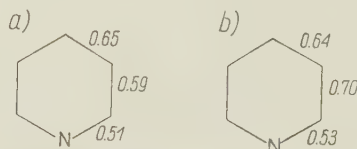


Fig. 5. Mobile bond orders in pyridine: a) present calculation, b) calculated by LCAO method [1]

in this work is in agreement with that resulting from the LCAO method. \*) Similar conformity exists also for bond orders with the exception of bonds 2,3. Fig. 4 shows the charge distribution for pyrimidine.

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\*) This agreement may be only qualitative, because the definitions of values responsible for an amount of charge in the environment of an atom both in the LCAO and FEMO methods are quite different.

# On the Conservation of Isofermions and the Rule $\Delta S = \Delta Q$ in Weak Processes

by

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*Presented by W. RUBINOWICZ on November 16, 1960*

It has been recognized for some time that in the domain of strong and electromagnetic interactions the conservation of strangeness  $\Delta S = 0$  is equivalent to the conservation of "isofermions"  $\Delta Y = 0$  (i.e. particles belonging to isodoublets [1]). The concept of strangeness may be formally extended also to leptons (cf. [2]—[4]), e.g. by means of the general Gell-Mann-Nishijima formula

$$(1) \quad Q = T_3 + \frac{N^B + S}{2},$$

where  $N^B$  is the number of baryons. This concept may be important for the classification of elementary particles and for the construction of their interactions; it does not seem, however, to provide us with any selection rule valid for *all* weak interactions including leptons. The purpose of the present paper is to discuss briefly a possibility of conservation of "isofermions" in the weak processes and, alternatively, of the rule  $\Delta S = \Delta Q$  for the strangeness changing part of the whole weak current.

It is well known that the baryons and mesons can be ordered as follows

$$(2) \quad \begin{array}{l} \left\{ \begin{array}{lll} \text{isodoublet } p, n & \frac{1}{2} & 0 & 1 \\ \text{isosinglet } \Lambda & 0 & -1 & 0 \\ \text{isotriplet } \Sigma^+, \Sigma^0, \Sigma^- & 1 & -1 & 0 \\ \text{isodoublet } \Xi^0, \Xi^- & \frac{1}{2} & -2 & -1 \end{array} \right. \end{array}$$

and

$$(3) \quad \begin{array}{l} \left\{ \begin{array}{lll} \text{isodoublet } K^+, K^0 & \frac{1}{2} & 1 & 1 \\ \text{isotriplet } \pi^+, \pi^0, \pi^- & 1 & 0 & 0 \\ \text{isodoublet } K^-, \bar{K}^0 & \frac{1}{2} & -1 & -1, \end{array} \right. \end{array}$$

where  $Y$  denotes the number of "isofermions" or (in other terms) the hypercharge. We have here

$$(4) \quad Y = N^B + S.$$

For photons we put  $S = 0$ ,  $Y = 0$ . Antiparticles have  $S$  and  $Y$  opposite to the corresponding particles.

Let us assume that for leptons we have the following scheme (provided there are two kinds of neutrinos,  $\nu$  and  $\nu'$ ):

$$(5) \quad \begin{array}{ccc} & T & S \quad Y \\ \left\{ \begin{array}{l} \text{isosinglet } \nu \\ \text{isodoublet } \nu', e^- \\ \text{isosinglet } \mu^- \end{array} \right. & \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & -1 & -1 \\ 0 & -2 & 0. \end{array} \end{array}$$

Scheme (4) as well as (2) follows from the theory described in [4], but in the present paper it can be considered as a phenomenological assumption. Let us remark that for leptons formula (4) is not valid.

We can see from (2), (3) and (5) that, assuming the conservation law of "isofermions",  $\Delta Y = 0$ , we exclude immediately the unobserved weak processes  $\mu^- \rightarrow e^- + \pi^0 + e^-$ ,  $\mu^- + N \rightarrow e^- + N$  and  $\pi^0 \rightarrow \mu^- + e^-$  without making use of any additional assumptions on the charge changing weak current. We forbid also the unobserved process  $\mu^- \rightarrow e^- + \gamma$ . The weak processes  $\mu^- \rightarrow e^- + \nu + \nu'$ ,  $\mu^- + p \rightarrow \nu' + n$ ,  $n \rightarrow p + e^- + \bar{\nu}$  and  $\pi^- \rightarrow \left\{ \begin{array}{l} \mu^- + \bar{\nu}' \\ e^- + \bar{\nu} \end{array} \right.$  are also forbidden by the conservation law  $\Delta Y = 0$ , whereas the weak processes  $\mu^- \rightarrow e^- + \nu' + \nu$ ,  $\mu^- + p \rightarrow \nu + n$ ,  $n \rightarrow p + e^- + \bar{\nu}'$  and  $\pi^- \rightarrow \left\{ \begin{array}{l} \mu^- + \nu \\ e^- + \nu' \end{array} \right.$  are allowed. Also, the unobserved electromagnetic or weak processes  $p \rightarrow \left\{ \begin{array}{l} p + e^- + e^+ \\ p + \nu'(\nu) + \bar{\nu}'(\nu) \end{array} \right.$ ,  $n \rightarrow \left\{ \begin{array}{l} n + e^- + e^+ \\ n + \nu'(\nu) + \bar{\nu}'(\nu) \end{array} \right.$  (in the nuclear matter) and  $\pi^0 \rightarrow \left\{ \begin{array}{l} e^- + e^+ \\ \nu'(\nu) + \bar{\nu}'(\nu) \end{array} \right.$  are not forbidden by the law  $\Delta Y = 0$  alone. They are, however, dominated by the electromagnetic processes  $p \rightarrow p + \gamma$ ,  $n \rightarrow n + \gamma$  (in the nuclear matter) and  $\pi^0 \rightarrow 2\gamma$ , respectively.

Assuming, on the contrary, that the lepton weak current is strangeness and charge changing according to the rule  $\Delta S = \Delta Q$  we cancel the conservation law  $\Delta Y = 0$ , nevertheless all unwanted weak processes are excluded. The allowed processes are  $\mu^- \rightarrow e^- + \nu + \nu'$ ,  $\mu^- + p \rightarrow \nu' + n$ ,  $n \rightarrow p + e^- + \bar{\nu}$  and  $\pi^- \rightarrow \left\{ \begin{array}{l} \mu^- + \bar{\nu}' \\ e^- + \bar{\nu} \end{array} \right.$ , whereas  $\mu^- \rightarrow e^- + \nu' + \nu$ ,  $\mu^- + p \rightarrow \nu + n$ ,  $n \rightarrow p + e^- + \bar{\nu}'$  and  $\pi^- \rightarrow \left\{ \begin{array}{l} \mu^- + \bar{\nu} \\ e^- + \nu' \end{array} \right.$  are forbidden.

It is well known that the rule  $\Delta S = \Delta Q$  applied to the strangeness changing part of the baryon and meson weak current excludes all unwanted leptonic decays of hyperons and kaons. In this case the rule  $\Delta S = \Delta Q$  applied as before to the lepton weak current is consistent with the older rule. Thus, the rule  $\Delta S = \Delta Q$  may be of general validity for the strangeness changing part of the whole weak current. In this case, the conservation law  $\Delta Y = 0$  could not be satisfied.

If one wants, on the contrary, to extend the conservation law  $\Delta Y = 0$  to hyperon and kaon decays, one must assume some "schizon" isospin properties for hyperons

and kaons. For instance, one may conjecture that (2) and (3) are proper for the strong and electromagnetic interactions, whereas in the weak interactions the hyperons  $\Lambda$ ,  $\Sigma^+$ ,  $\Sigma^0$ ,  $\Sigma^-$ , group themselves rather in two Gell-Mann's isodoublets  $Y$  and  $Z$ , and the kaons  $K^+$ ,  $K^0$ ,  $\bar{K}^0$ ,  $K^-$  in one isosinglet  $K_1^0$  and one isotriplet  $K^+$ ,  $K_2^0$ ,  $K^-$  containing the Gell-Mann-Pais neutrals  $K_1^0$  and  $K_2^0$ . Thus, for weak interactions we would have

$$(6) \quad \left\{ \begin{array}{lll} p, & n & T \quad S \quad Y \\ Y^+ = \Sigma^+, Y^0 = \frac{1}{\sqrt{2}}(\Lambda^0 - \Sigma^0) & \frac{1}{2} & 0 \quad 1 \\ Z^0 = \frac{1}{\sqrt{2}}(\Lambda^0 + \Sigma^0), Z^- = \Sigma^- & \frac{1}{2} & 0 \quad 1 \\ \Xi^0, & \Xi^- & 1 \quad 0 \quad -1 \end{array} \right.$$

and

$$(7) \quad \left\{ \begin{array}{lll} \pi^+, \pi^0, \pi^- & T \quad S \quad Y \\ K_1^0 = \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0) & 1 \quad 0 \quad 0 \\ K^+, K_2^0 = \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0), K^- & 0 \quad 0 \quad 0 \\ & 1 \quad 0 \quad 0. \end{array} \right.$$

Making use of (6), (7) and (5) we can see that the conservation law  $\Delta Y = 0$  allows, e.g., the weak processes  $\Lambda \rightarrow \begin{cases} p + \pi^- \\ n + \pi^0 \end{cases}$ ,  $\Lambda \rightarrow \begin{cases} p + e^- + \bar{\nu}' \\ n + \mu^- + \bar{\nu} \end{cases}$ ,  $K^- \rightarrow \begin{cases} \mu^- + \bar{\nu} \\ e^- + \bar{\nu}' \end{cases}$  and  $K \rightarrow \begin{cases} 2\pi \\ 3\pi \end{cases}$  but also the unobserved weak processes  $\Lambda \rightarrow \begin{cases} n + e^- + e^+ \\ n + \nu'(v) + \bar{\nu}'(v) \end{cases}$  and  $\bar{K}^0 \rightarrow \begin{cases} e^- + e^+ \\ \nu'(v) + \bar{\nu}'(v) \end{cases}$  are not forbidden by the law  $\Delta Y = 0$  alone and are not dominated here by the processes  $\Lambda \rightarrow n + \gamma$  and  $K^0 \rightarrow 2\gamma$ , respectively. The unobserved weak process  $\bar{K}^0 \rightarrow \mu^- + e^+$  is forbidden.

Concluding, the general rule  $\Delta S = \Delta Q$  applied to the strangeness changing part of the whole weak current seems to be more probable than the conservation rule  $\Delta Y = 0$ . Simple assumption of the rule  $\Delta S = \Delta Q$  for the strangeness changing part of the of the weak current does not help, of course, to explain the approximate rule  $|\Delta T| = \frac{1}{2}$  observed in pionic decays of hyperons and kaons.

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## On the Fluorescence of Fluorescein in Methyl Methacrylate

by

A. BĄCZYŃSKI, M. CZAJKOWSKI and S. TRAWIŃSKI

*Presented by A. JABŁOŃSKI on November 18, 1960*

Photoluminescence of fluorescein was observed in various media. Fluorescein in a liquid or solid medium was proved to occur as a negative or positive ion according to the kind of medium considered, and, in particular, according to its  $pH$ , [5], [11], [15]. Hence, the solvent and its  $pH$  strongly affect the position of the absorption and emission spectra of the dye. Moreover, the following was found to hold:

a) The long-wave absorption related to the transition of the valence electrons of the molecule as a whole to a nearest excited electronic state (principal absorption band) is greater than the absorption of radiation of shorter wavelengths due to transitions in aromatic radicals and rings [19], [20] and to transitions from the fundamental level to energy states higher than the first excited state ( $X' Y'$  of [10]).

b) The maxima of the absorption and emission spectra of fluorescein in an acid medium, e.g. boric acid, lie approximatively at 4300 and 4800 Å, respectively [15]. In alkaline media, such as glycerine, water with an admixture of ammonia, or ethyl alcohol [2], [3], the maximum of absorption occurs at 4800 Å, whereas that of emission lies at 5400 Å. In weakly acid or weakly alkaline media, both bands are sometimes found to occur simultaneously.

In addition to the rigid solutions (boric acid, glucose, collodium, aluminium sulphate etc.) and the various liquid solutions used hitherto, macromolecular compounds are now increasingly applied, e.g., methyl methacrylate, which transforms into a vitreous rigid medium in the process of polymerisation. The present paper deals with the absorption and emission spectra of fluorescein in methyl methacrylate, with the influence of alcohol on the positions of the maxima of bands and with polarisation of fluorescein fluorescence at various concentrations.

The absorption spectrum of fluorescein in methyl methacrylate (Fig. 1) is characterized by the absence of a distinct band within the long-wave region of absorption where, in the media most commonly used, the principal band occurs. At most, traces of a band are perceptible. The maximum of the absorption band lies essentially outside the spectral region comprised by the Zeiss spectrophotometer used in measuring the absorption spectra. Experimental conditions provided for recording of absorption spectra extending from 3500 Å to the other extremity of the visible

spectrum. From Fig. 2, the emission spectrum is clearly seen to consist of the superposition of several bands, some of which could possibly be due to the positive ions and others to negative ions of the dye (clear maxima about 4800 and 5050 Å and humps pointing to the possible presence of a characteristic band for fluorescein in an alkaline medium).

Methyl methacrylate is a very bad solvent for fluorescein whose maximum concentration therein is of the order of  $10^{-4}$  g/ml. The solubility of fluorescein in methyl methacrylate can be raised by adding ethyl alcohol to the medium [8]. The effect of alcohol on the emission spectra of acetylanthracenes in hexane was



Fig. 1. Absorption spectra of fluorescein in methyl methacrylate monomer  
Curve A — in pure monomer, curve B — in binary solution (90 per cent methyl methacrylate, 10 per cent ethyl alcohol).

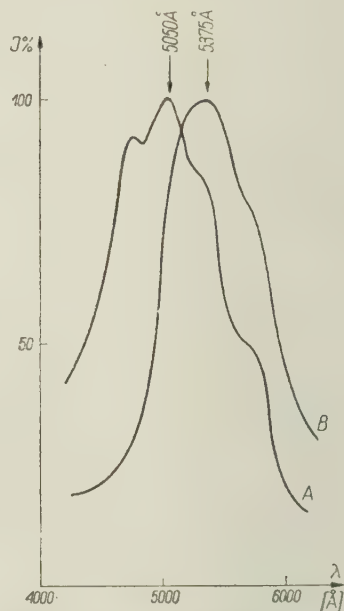


Fig. 2. Emission spectra of fluorescein in methyl methacrylate monomer  
Curve A — in pure monomer, graph B — in binary solution (90 per cent methyl methacrylate and 10 per cent ethyl alcohol)

investigated by Tcherkasov [4], who found that alcohol produces an appreciable shift of the maxima towards longer wavelengths. In the case of fluorescein in methyl methacrylate containing alcohol, the emission spectra are shifted towards longer wavelengths (Fig. 2), whereas the absorption spectrum now exhibits a wide band within the long-wave region (Fig. 1). Since the absorption and emission bands of fluorescein in methyl methacrylate with alcohol approximately coincide with those of fluorescein in ethyl alcohol, it seems reasonable that the nearest neighbourhood of a molecule of the dye is occupied by molecules of the alcohol [16], or that associates of fluorescein and ethyl alcohol are formed.

Polymerisation of methyl methacrylate produces essential changes of positions and shapes of the absorption bands, causing a shift of the maxima towards longer wavelengths (Fig. 3); it does not, however, affect the position of the emission spectra. In the conditions of the present investigation, polymerisation proceeds at a slow rate in vessels which provide for a constant concentration of dye and alcohol.

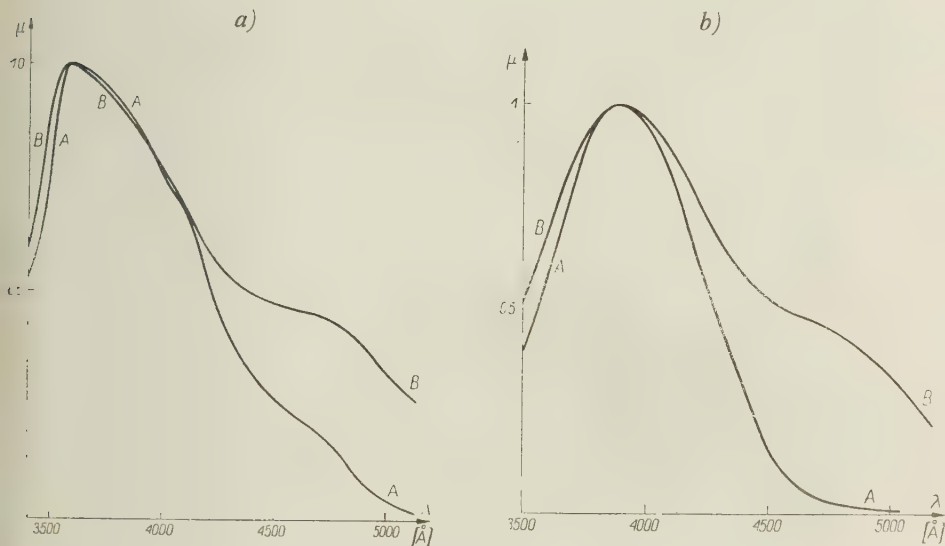


Fig. 3. a) Absorption spectra of fluorescein in methyl polymethacrylate

A — fluorescein concentration  $c = 5 \times 10^{-5}$  g/ml, B — fluorescein concentration  $c = 10^{-4}$  g/ml

b) Absorption spectra of fluorescein in methyl polymethacrylate containing 10 per cent of ethyl alcohol

A — fluorescein concentration  $c = 5 \times 10^{-6}$  g/ml, B — fluorescein concentration  $c = 10^{-3}$  g/ml.

The considerable differences between the absorption as well as emission spectra of fluorescein in methyl methacrylate monomer with alcohol content and those in the pure monomer, are almost levelled down in long-wave region of the absorption spectrum by the process of polymerisation of methacrylate with alcohol. The insignificant differences in the positions of the absorption maxima of either spectrum may result from differences in the degree of polymerisation of the methacrylate. The above results would make it seem plausible that during polymerisation the alcohol forms a compound with methyl methacrylate, usually occupying the sites at an extremity of the polymer chain (alcohol acts as a degrading factor [9]), whereas the associates of dye and alcohol, or, maybe, the solvation shells consisting of molecules of alcohol surrounding those of fluorescein are destroyed.

The degree of polarisation of fluorescein fluorescence *versus* the concentration of dye molecules was determined in the concentration range of  $5 \times 10^{-6}$  to  $10^{-3}$  g/ml for methyl polymethacrylate with alcohol, and of  $5 \times 10^{-6}$  to  $10^{-4}$  g/ml for that

with no alcohol. The degree of polarisation was found to be independent of the dye concentration and of the alcohol content. The measurements were carried out by the photoelectric method proposed by Wille [21]. Light transmitted through a Zeiss UG 1/3.5 filter was used for excitation. An OG 1/1 filter was placed before the photomultiplier.

The results obtained can be summarised as follows. The effect of alcohol quite distinct in methyl methacrylate monomer is absent in the case of rigid (polymerized) solutions. Moreover, although the maxima of the absorption and emission bands are wide apart, there is considerable superposition of their wings (Fig. 4).\*) The lack of self-depolarisation as found by Kawski and co-workers [7], [8] and by the present authors thus remains unexplained (at the concentration of  $10^{-3}$  g/ml, some small decrease in the degree of polarisation was observed to occur).

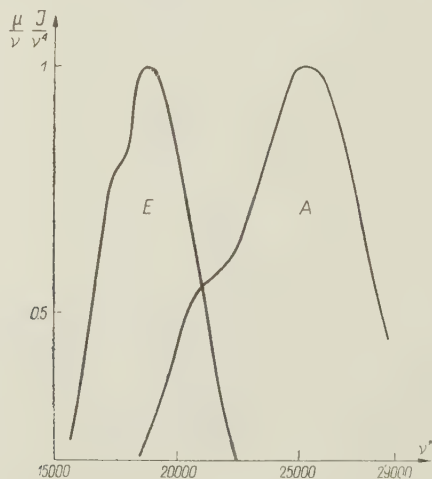


Fig. 4. Absorption and emission spectrum of fluorescein in methyl polymethacrylate.

On comparing the absorption curves for different fluorescein concentrations, on which the main maxima are made equal, both for methyl methacrylate with and without alcohol (Figs. 3a, 3b), a second band depending solely on the dye concentration but independent of the alcohol introduced into the medium is found to develop within the long-wave spectral region. The formation of a second band in the absorption spectrum, the intensity of which depends on the concentration of the dye, is usually ascribed to the formation of associates of dye molecules (see,

\*) In order to obtain a useful scale of comparison, the intensity of the light emitted has been divided by  $\nu^4$  and the coefficient of absorption by  $\nu$ . The choice of such units results immediately from the Einstein coefficients determining the probability of emission and absorption. This manner of comparison of the emission and absorption spectra is adopted by various authors (see, i.a. [1], [2], [17], [18]).

i.a. [6], [12], [13]). However, it cannot be ruled out that the change of  $pH$  of the solution due to changes in the dye concentration [14] is the cause of the appearance of this band.

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## Yield of Anti-Stokes Fluorescence of Chlorophyll

by

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*Presented by A. JABŁOŃSKI on November 24, 1960*

The course of the efficiency drop of chlorophyll fluorescence in anti-Stokes excitation region is of particular importance in view of the interdependence of the phenomenon of fluorescence and that of photosynthesis. J. Franck [1], for instance, has put forward a common explanation for the occurrence of a long-wave boundary of fluorescence yield and photosynthesis for chlorophyll *a* from green and green-blue algae.

The knowledge of the dependence of fluorescence yield on the wavelength of the exciting light in the anti-Stokes excitation region enables us to draw some conclusions concerning the aggregation state of molecules in a given solution [2]—[4].

Since the fluorescence yield of chlorophyll *a* in this excitation region has so far been measured only at a few points [5] (marked on the Figure by crosses), the present authors have measured the course of fluorescence yield as accurately as possible. Ether was chosen as a solvent, since Foster and Livingstone [5] have already measured the absolute yields of fluorescence for chlorophyll *a* in this solvent which makes possible computation of the absolute efficiency values from the relative ones.

The fluorescence yields obtained are presented in the Figure, together with absorption- and fluorescence bands measured for the same solution.

As may be seen from the Figure, the efficiency drop occurring mostly for anti-Stokes excitations of dye solutions begins in the neighbourhood of the maximum absorption-band (6610 Å). The yield for shorter waves is almost constant, the minimum yield, however, may be observed close to ca. 6250 Å. The appearance of this minimum is in agreement with previous measurements [5], which are plotted on the same figure for comparison. The measuring device and the method of isolating chlorophyll *a* were the same as described in our earlier paper [6].

The relative yield of fluorescence  $\eta(\lambda)/\eta(\lambda_0)$  was calculated from the expression

$$\frac{\eta(\lambda)}{\eta(\lambda_0)} = \frac{I(\lambda_0) \cdot F(\lambda)}{I(\lambda) \cdot F(\lambda_0)} \cdot \frac{\exp[-\mu(\lambda_0)x_1] - \exp[-\mu(\lambda_0)x_2]}{\exp[-\mu(\lambda)x_1] - \exp[-\mu(\lambda)x_2]},$$

where:

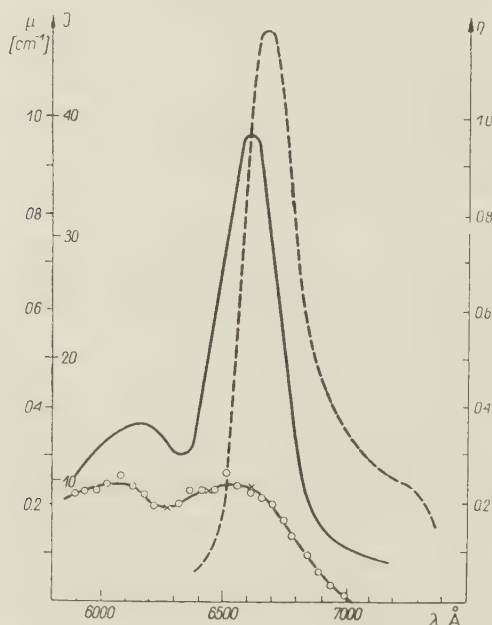
$\eta(\lambda)$ ,  $\eta(\lambda_0)$  — denote absolute yields of fluorescence for the investigated and standard wave-lengths of exciting light, respectively;

$I(\lambda)$ ,  $I(\lambda_0)$  — intensities of light emitted from the monochromator for these wave-lengths;

$F(\lambda)$ ,  $F(\lambda_0)$  — intensities of fluorescence light "seen" by the photomultiplier placed perpendicularly to the direction of the exciting beam;

$\mu(\lambda)$ ,  $\mu(\lambda_0)$  — absorption coefficients;

$x_1$ ,  $x_2$  — distance of the two edges of an additional slot in the diaphragm (placed in these measurements before the container with the solution), from the inlet of the excitation beam into the solution.



Absorption- fluorescence- and yield-spectra of chlorophyll in ether

Yield of fluorescence: ○ — author's measurements, x — measurements of Foster and Livingstone [5]

————— absorption band, - - - - - fluorescence band

The values  $F(\lambda)$ ,  $F(\lambda_0)$ ,  $\mu(\lambda)$ ,  $\mu(\lambda_0)$  as well as the fluorescence spectra were determined by the arrangement described in [6]; for  $F(\lambda)$  measurements the container with the solution was shielded by diaphragm with a slot  $X_2 - X_1$  wide.

$I(\lambda)$  and  $I(\lambda_0)$  were measured by means of a thermopile.

The experimental error of  $\eta$  varies at various measurement points, it does not, however, exceed 20 per cent.

The determination of the total curve of  $\eta(\lambda)$  for the ether solution makes it possible to determine the courses of absolute yield for chlorophyll solutions in other solvents using the method described in [7].

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# БЮЛЛЕТЕНЬ ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ  
И ФИЗИЧЕСКИХ НАУК

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А. СУЛИНСКИЙ, КЛАССИФИКАЦИЯ ПОЛУПРОСТЫХ КОЛЕЦ

стр. 1—6

Пусть  $\mathfrak{r}$  — полупростое кольцо в смысле радикала Брауна и Мак-Коя, а  $M$  — пространство всех его максимальных модулярных идеалов. Через  $I(N)$ , где  $N \subseteq M$ , обозначим пересечение всех идеалов  $\mathfrak{m} \in N$ . Замыканием  $\bar{N}$  множества  $N$  называем множество всех таких  $\mathfrak{m} \in M$ , что  $\mathfrak{m} \supseteq I(N)$ . Положим  $D = M - M'$ , где  $M'$  есть множество всех таких  $\mathfrak{m} \in M$ , что  $\mathfrak{m} \in \overline{M - (\mathfrak{m})}$ . Полупростое кольцо  $\mathfrak{r}$  называется специальным, если  $\bar{D} = M$ ; если же  $D$  — пустое, то  $\mathfrak{r}$  будем называть вполне неспециальным. Идеал  $s(\mathfrak{r}) = I(M - \bar{D})$  называется специальной частью а идеал  $n(\mathfrak{r}) = I(D)$  — вполне неспециальной частью кольца  $\mathfrak{r}$ . Доказывается, что кольцо  $\mathfrak{r}/n(\mathfrak{r})$  специально, а кольцо  $\mathfrak{r}/s(\mathfrak{r})$  вполне неспециально. Кроме того, если  $\mathfrak{r}$  — сильно полупростое кольцо (т. е. если всякий идеал  $\mathfrak{a}$  в  $\mathfrak{r}$  представляется в виде  $\mathfrak{a} = I(N)$ ), то  $I(\text{Fr } \bar{D}) = s(\mathfrak{r}) \dot{+} n(\mathfrak{r})$ , где  $\text{Fr } D = \bar{D} \cap M - \bar{D}$ .

Пусть  $D_1 = D$ ,  $M_0 = M$ ,  $M_1 = \text{Fr } \bar{D}_1$ . Положим  $M_\alpha = D_\alpha \cap M_{\alpha-1} - D_\alpha$ , где  $D_\alpha = M_\alpha - M'_\alpha$ , если существует порядковое число  $\alpha = 1$ . Если же  $\alpha$  — предельное число, то  $M_\alpha = \bigcap_{\xi < \alpha} M_\xi$ . Наименьшее порядковое число  $\tau$  такое, что  $M_{\tau+1} = M_\tau$ , назовем типом кольца  $\mathfrak{r}$ . Доказывается, что если  $\mathfrak{r}$  сильно полупростое кольцо типа  $\tau$ , то  $s_\alpha = s(c_\alpha) \dot{+} n(c_\alpha)$ , где  $s_\alpha = I(M_{\alpha+1}) \cdot I(M_\alpha)$  и  $\alpha < \tau$ . Кроме того, для любого  $\alpha < \tau$  существует возрастающая последовательность типа  $\tau$  —  $\alpha$  специальных колец.

$$s(c_\alpha) = s_{\alpha+1}(c_\alpha) \subset \dots \subset s_\xi(c_\alpha) \subset \dots \quad (\alpha < \xi \leq \tau),$$

и такая же последовательность вполне неспециальных колец.

$$n(c_\alpha) = n_{\alpha+1}(c_\alpha) \subset \dots \subset n_\xi(c_\alpha) \subset \dots \quad (\alpha < \xi \leq \tau)$$

причем

$$s_{\xi+1}(c_\alpha) \cdot s_\xi(c_\alpha) \sim n_{\xi+1}(c_\alpha) / n_\xi(c_\alpha) \sim c_\xi.$$

К. МОРЭН, **ОТОБРАЖЕНИЯ ТИПА ГИЛЬБЕРТА-ШМИДТА И ИХ ПРИМЕНЕНИЕ К РАЗЛОЖЕНИЯМ ПО СОБСТВЕННЫМ ФУНКЦИЯМ И К ЭЛЛИПТИЧЕСКИМ КРАЕВЫМ ЗАДАЧАМ** . . . . . стр. 7—11

Пусть  $E, F$  — сепарабельные пространства Гильберта и  $(e_i)_1^\infty$  — полная ортонормальная система в  $E$ . Линейное отображение  $A: E \rightarrow F$  называется отображением Гильберта-Шмидта (отображ. Г.-Ш.) если  $\sum \|Ae_i\|_F^2 < \infty$ . Оказывается, что (о. Г.-Ш.) является непрерывным. Суперпозиция непрерывных отображений из которых, по крайней мере, одно — о. Г.-Ш., является тоже отображением Гильберта-Шмидта; суперпозиция отображений Гильберта-Шмидта является уже ядерным отображением. Вложение Соболева

$$H^{m+k}(\Omega_N) \rightarrow H^k(\Omega_N), \text{ где } m > N/2, k \geq 0$$

является отображением Гильберта-Шмидта.

Отсюда тотчас же вытекает обобщение теоремы Браудера: Если  $R$  — резольвента (в  $L^2(\Omega_N)$ ) корректной эллиптической задачи порядка  $r$ , тогда  $R^j$  является оператором Гильберта-Шмидта для  $jr > N/2$ .

Пусть  $F: H \rightarrow \hat{H} = \int_A \hat{H}(\lambda) d\mu(\lambda)$  — обобщенное отображение Фурье.

**ОСНОВНАЯ ТЕОРЕМА:** Если  $\Phi \subset H$  и вложение  $\Phi \rightarrow H$  является о. Г.-Ш., тогда  $F(\lambda): \Omega \ni \varphi \rightarrow \hat{\varphi}(\lambda) \in \hat{H}(\lambda)$  является почти для всех  $\lambda \in I$  отображением типа Гильберта-Шмидта.

Из основной теоремы вытекают тотчас же обобщения спектральных теорем Гельфанда-Костюченко, Березанского, Гординга и других.

Д. ЗАРЕМБА, **О ЗАКРЕПЛЕНИИ И НАНИЗЫВАНИИ КОМПАКТНЫХ МНОЖЕСТВ** . . . . . стр. 13—15

Пусть  $E$  — метрическое пространство. Множество  $X \subset E$  закреплено в  $E$  (см. Б. Кнастер [1]), если для всякого  $\varepsilon > 0$  существует в  $E$  конечная сумма  $F_\varepsilon = F_1 \cup F_2 \cup \dots \cup F_{k(\varepsilon)}$  непересекающихся замкнутых множеств диаметра  $\delta(F_i) < \varepsilon$ , которая пересекает все компоненты множества  $X$ . Оно же называется *нанизываемым*, если в  $E$  имеется простая дуга  $L$ , пересекающая все компоненты множества  $X$ .

В работе доказываются теоремы, по которым нанизываемость компакта  $X$  в евклидовом  $n$ -мерном пространстве влечет за собой его закрепляемость, а в  $E^2$  — эквивалентна ей. Подробности, примеры и другие полученные результаты войдут в работу, написанную автором для Fundamenta Mathematicae.

А. ЭРЕНФОЙХТ, **СЕПАРАБЕЛЬНЫЕ ТЕОРИИ** . . . . . стр. 17—19

В работе приводится анализ связи между существенной неразрешимостью формализованных теорий и сепарабельностью множества тезисом теории от множества отрицаний тезисов.

В первой части работы содержится следующая теорема:

*Аксиоматизируемая теория  $T$  — несепарабельна тогда и только тогда, когда для каждого рекурсивного семейства  $\{T_i\}$   $i = 1, 2, \dots$ , ее аксиоматизируемых расширений имеется предложение  $\Phi$ , неразрешимое в каждой из теорий  $T_i$ .*

Во второй части работы дается пример теории существенно неразрешимой, в которой множество тезисов и множество отрицаний тезисов можно отделить при помощи рекурсивных множеств.

ХИ ГУАН-ФУ, **О СИЛЬНОЙ И СЛАБОЙ СХОДИМОСТИ ОСОБЫХ ИНТЕГРАЛОВ ВЕКТОРНЫХ ФУНКЦИЙ** . . . . . стр. 21—27

Автор доказывает уравнение при предположениях (1), (2), (3) относительно ядра  $K_n(s, t)$  и при предположении, что в пространстве Банаха  $X$  слабая сходимость не равносильна (не эквивалентна стандартной сходимости) имеется ограниченная и интегрируемая в смысле Бохнера векторная функция с значениями из  $X$  такая, что особые интегралы

$$\Phi_n(x, t) = \int_a^b x(s) K_n(s, t) dt$$

в данном счетном множестве  $RC(a, b)$  сходятся к  $x(t)$  слабо, но не сильно, а в  $(a, b)$  —  $R$  сходятся сильно к  $x(t)$ .

Ю. РАЙСКИЙ, **ЗАКОНЫ СОХРАНЕНИЯ В ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ** . . . . . стр. 33—37

Дается формулировка законов сохранения момента энергии и углового момента в общей теории относительности при использовании тетраэдрической техники.

Приведенная формулировка удовлетворяет всем требованиям точной определимости положения в общей ковариантности.

В качестве примера физической интерпретации дается определение класса четырехгранных полей в римановской геометрии, отвечающее определению постоянных полей в евклидовой геометрии.

В. ВОЗЬНИЦКИЙ и С. КВЯТКОВСКИЙ, **ОЦЕНКА ВЛИЯНИЯ ПОСТОРОННИХ АТОМОВ В СКЕЛЕТЕ СОПРЯЖЕННЫХ СВЯЗЕЙ НА СПЕКТР И РАСПРЕДЕЛЕНИЕ ЗАРЯДА В МОЛЕКУЛЕ** . . . . . стр. 39—42

Проблема оценки возмущающего влияния, которое оказывают посторонние атомы или группы в скелете сопряженных связей, рассматривалась до настоящего времени на базисе метода LCAO MO. Пред-

принята попытка разъяснения этой проблемы для молекул пиридина и пиримидина на основании метода ФЕМО путем введения потенциала, ответственного за возмущение, произведенное атомом азота.

Результаты полученные для спектра поглощения, для распределения электронного заряда и для порядка связей находятся в согласии с результатами, полученными при применении других методов, а также в согласии с проведенными экспериментами.

**В КРУЛИКОВСКИЙ, О ПОВЕДЕНИИ ИЗОФЕРМИОНОВ И О ПРАВИЛЕ  $\Delta S = \Delta Q$  В СЛАБЫХ ВОЗДЕЙСТВИЯХ** . . . . . стр. 43—45

В работе дискутируется возможность сохранения изофермионов в слабых взаимодействиях или альтернативного правила  $\Delta S = \Delta Q$  для слабого тока, изменяющего „strangeness“.

**А. БОНЧИНСКИЙ, М. ЧАЙКОВСКИЙ и С. ТРАВИНСКИЙ, О ФОТОЛЮМИНЕСЦЕНЦИИ ФЛУОРОСЦЕИНА В МЕТАКРИЛАНЕ МЕТИЛА** стр. 47—51

Исследованы спектры поглощения и эмиссионные спектры флуоресцеина в мономере и полимере чистого метакрилана метила, а также с 10%-ым добавлением этилового спирта. .

Констатировано отчетливое влияние этилового спирта на спектры поглощения флуоресцеина в мономере метакрилана метила, тогда как в полимере это влияние отчетливо исчезает.

Авторы сугеррируют, что наблюдаемые изменения спектров поглощения (образование длинноволновой полосы поглощения в мономере метакрилана метила в присутствии спирта) вызваны локализацией молекул этилового спирта в ближайшей среде молекул красителя или же образованием ассоциированных соединений молекул красителя со спиртом. В процессе полимеризации спирт вступает в реакцию (реагирует) с полимерной цепью и вследствие этого сольватная оболочка спирта вокруг молекул красителя или же ассоциированное соединение этилового спирта с молекулой красителя, подвергается разрушению.

**Д. ФРОНЦОВЯК и Т. МАРШАЛЭК, ВЫХОД АНТИСТОКСОВСКОЙ ФЛУОРЕСЦЕНЦИИ ХЛОРОФИЛЛА** . . . . . стр. 53—55

Авторами был измерен с возможной точностью пробег выхода флуоресценции хлорофилла „а“ в эфире в зависимости от длины волны возбуждающего света в области от 6000Å до 7040Å.

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Cena zł 20.—